Advanced Calculus II - 2022/23

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Victory won't come to us unless we go to it.

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Tutorial 1 2022/9/21

1.1 Riemann integral

We first recall the definition of Riemann integrable:

Definition 1.1

A function $f : [a, b] \times [c, d] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if there exists a number I, such that for any $\epsilon > 0$, there exists $\delta > 0$, and for any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b, c = y_0 < y_1 < \cdots < y_m = d$ such that $\Delta x_i := x_{i+1} - x_i < \delta, \Delta y_j < \delta$ and for any tags $\tau = \{p_{ij}, p_{ij} \in [x_i, x_{i+1}] \times [y_j, y_{j+1}]\}$, we have the Riemann sum

$$S(f, P, \tau) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} f(p_{ij}) \Delta x_i \Delta y_j$$

such that

$$|S(f, P, \tau) - I| < \epsilon.$$

♣

The definition adapts to arbitrary dimension by changing the number of parameters.

Problem 1.1 Let

$$\phi(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$$

show that $\phi(x)$ is not Riemann integrable on $[0, 1]$.

Proof Assume it is integrable, then we have $\exists I, \forall \epsilon > 0, \exists \delta > 0, \forall \mathcal{P}$ partition and $\forall \tau$ tags on $\mathcal{P}$, the Riemann sum $S(\phi, P, \tau) = \sum_{i=0}^{n-1} f(p_i) \Delta x_i$ satisfies the inequality

$$|S(f, P, \tau) - I| < \epsilon.$$

Then we have

$$\frac{1}{p_0} x_1 = f(p_0) \Delta x_0 < -\sum_{i=1}^{n-1} f(p_i) \Delta x_i + \epsilon + I.$$

This holds for any p_0 such that $0 < p_0 < x_1$. Let p_0 tend to 0, then $f(p_0) \Delta x_0$ can not be bounded as above, which is a contradiction.

Remark In fact, you can use the same argument to show that if a function is not bounded on some interval, then it is not integrable.

Problem 1.2 Let

$$g(x, y) = \begin{cases} 1 & x, y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

show that g is not Riemann integrable on $R = [a, b] \times [c, d]$.

Proof Let P be any partition of the rectangle. By choosing tags points $p_{ij}\left(x_i^*, y_j^*\right)$ where x_i^* and y_j^* are rational numbers,

$$\sum_{i,j} g\left(x_i^*, y_j^*\right) \Delta x_i \Delta y_j = \sum_{i,j} \Delta x_i \Delta y_j$$

which is equal to the area of R. On the other hand, by choosing the tags so that x_i^* is irrational, $g\left(x_i^*, y_j^*\right) = 0$

so that

$$\sum_{i,j} g\left(x_i^*, y_j^*\right) \Delta x_i \Delta y_j = \sum_{i,j} 0 \times \Delta x_i \Delta y_j = 0.$$

Depending the choice of tags, the Riemann sums are not the same for the same partition, hence they cannot tend to the same limit no matter how small their norms are. We conclude that g is not integrable.

Remark You can conclude that

$$g_n(x_1, x_2, \cdots, x_n) = \begin{cases} 1 & \forall i, x_i \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is not integrable

1.2 Henstock–Kurzweil integral

We see that many functions are not integrable (For example as long as it is not bounded), but in many cases they would have calculable area below the graph, such as $\int_0^1 \frac{1}{\sqrt{x}} \, \mathrm{d}\, x = 2$ where $\frac{1}{\sqrt{x}}$ is not integrable on $[0, 1]$. Also we would expect the integration of g_n on any rectangle to be 0 as the size of the set of rational numbers is way too smaller than that of the set of irrational numbers.

One way to make an adjustment is by Lebesgue's theory on integration which you will learn in a course real analysis. Another simpler way is what I will show you below.

Definition 1.2

A function $f : [a, b] \to \mathbb{R}$ is HK-integrable on $[a, b]$ if there exists a number I, such that for any $\epsilon > 0$, there exists a positive function $\delta : [a, b] \to (0, \infty)$, and for any partition $\mathcal{P} : a = x_0 < x_1 < \cdots < x_n = b$ and for any tags $\tau = \{p_i, p_i \in [x_i, x_{i+1}]\}$ such that $x_{i+1} - p_i < \delta(p_i)$ and $p_i - x_i < \delta(p_i)$, we have

$$|S(f, P, \tau) - I| < \epsilon.$$

♣

Note you can get this definition generalized to arbitrary dimension in the manner as we define the Riemann integrable in dimension two. The positive function δ is what the difference from Riemann integrable. In fact, a contant δ in the definition of Riemann integrable could be viewed as a function on $[a, b]$, so Riemann integrable imples HK-integrable.

Problem 1.3 Let

$$g(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

show that g is HK-integrable on $[0, 1]$ with integration $I = 0$.

Proof We list all rational numbers in $[0, 1]$ as $\{q_1, q_2, \cdots, q_n, \cdots\}$. For $\epsilon > 0$, we choose a delta

$$\delta(x) = \begin{cases} \frac{1}{2^{n+1}}\epsilon & x = q_n \\ 1/2 & \text{otherwise} \end{cases}$$

Consider the Riemann sum $S(g, P, \tau) = \sum g(p_i)\Delta x_i$. If p_i equals to some rational number q_n, then $\Delta x_i = x_{i+1} - x_i = x_{i+1} - q_n + q_n - x_i < 2\delta(q_n) = \frac{1}{2^n}\epsilon$. So we have

$$0 \le S(g, P, \tau) = \sum g(p_i)\Delta x_i \le \sum g(q_n)\frac{1}{2^n}\epsilon = \epsilon$$

Therefore the integration is 0.

Problem 1.4 Let

$$\phi(x) = \begin{cases} \frac{1}{\sqrt{x}} & x > 0 \\ 0 & x = 0 \end{cases}$$

show that $\phi(x)$ is HK-integrable on $[0,1]$.

Proof we shall show that $\int_0^1 f(x)dx = 2$. Let any $\varepsilon > 0$ be given; we are to exhibit a coresponding function δ. Let $\delta(0) = \frac{1}{16}\varepsilon^2$, and for each $x > 0$ choose $\delta(x) > 0$ small enough so that

$$[u,v] \subseteq (x-\delta(x), x+\delta(x)) \Longrightarrow \left|\frac{2}{\sqrt{u}+\sqrt{v}} - \frac{1}{\sqrt{x}}\right| < \frac{\varepsilon}{2}.$$

It follows easily that $\left|\left(2\sqrt{x_i} - 2\sqrt{x_{i-1}}\right) - f\left(p_{i-1}\right)\left(x_i - x_{i-1}\right)\right|$ is less than $\frac{1}{2}\left(x_i - x_{i-1}\right)\varepsilon$ when $p_{i-1} \neq 0$, or less than $\frac{1}{2}\varepsilon$ when $p_{i-1} = 0$. From this we obtain $\left|2 - \sum_{i=1}^N f\left(p_{i-1}\right)\left(x_i - x_{i-1}\right)\right| < \varepsilon$. The preceding argument proves the existence of a suitable δ, but it does not provide an explicit formula for δ. One choice that will work is

$$\delta(x) = \min\left\{\frac{1}{2}x, \frac{1}{4}x^{3/2}\varepsilon\right\} \qquad \text{when } x > 0.$$

To see that this will work, reason as follows: If $|x-u| < \delta(x)$, then $u > \frac{1}{2}$ x, hence

$$\sqrt{xu}(\sqrt{u}+\sqrt{x}) > x^{3/2}\sqrt{\frac{1}{2}}\left(\sqrt{\frac{1}{2}}+1\right) > \frac{x^{3/2}}{2}$$

hence

$$\left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{u}}\right| = \left|\frac{u-x}{\sqrt{xu}(\sqrt{u}+\sqrt{x})}\right| < \frac{\delta(x)}{\frac{1}{2}x^{3/2}} \leq \frac{1}{2}\varepsilon.$$

Thus $\frac{1}{\sqrt{u}}$ lies in $\left(\frac{1}{\sqrt{x}} - \frac{\varepsilon}{2}, \frac{1}{\sqrt{x}} + \frac{\varepsilon}{2}\right)$. The same reasoning shows that $\frac{1}{\sqrt{v}}$ also lies in that interval, if $|x-v| < \delta(x)$. Finally, $\frac{2}{\sqrt{u}+\sqrt{v}}$ lies between $\frac{1}{\sqrt{u}}$ and $\frac{1}{\sqrt{v}}$, since $\frac{\sqrt{u}+\sqrt{v}}{2}$ lies between $\sqrt{u}$ and $\sqrt{v}$

Remark The intuition behind the definition of HK-integrable is to make the subinterval of the partition that contains a pathological tag as small as possible.

Tutorial 2 2022/9/28

2.1 Improper multiple integral[1]

Improper integral is applied when we want to integrate the function over some unbounded domain or integrate some unbounded function. The idea behind it is to use something finite to approach something infinite, and likewise use bounded subsets to approach unbounded subsets.

Let's begin with the definition for improper multiple integral, but firstly we need to define some notions from topology to make the definition coherent and rigorous.

Definition 2.1

Let E be a subset of $\mathbb{R}^m$, E is bounded if there is a number R such that for all $x = (x_1, \cdots, x_m) \in E$ we have $\sum_{i=1}^m x_i^2 \leq R$. ♣

Definition 2.2

The closure of $E \subset \mathbb{R}^m$ is the subset $\{x \in \mathbb{R}^m | \exists x_n \in E, \lim_{n\to\infty} x_n = x\}$, denoted by $\bar{E}$. ♣

Definition 2.3

The boundary ∂E of E is the intersection of closure of E and the closure of the complement of E. That is, $\partial E = \bar{E} \cap \bar{E^c}$. ♣

Example 2.1 The closure of an open interval (a, b) is the closed interval $[a, b]$. The boundary of (a, b) is the set of the two endpoints $\{a, b\}$

In the following we write $\mathrm{d}\, x_1 \,\mathrm{d}\, x_2 \cdots \mathrm{d}\, x_m$ as $\mathrm{d}\, x$ for short.

Definition 2.4

A subset $E \subset \mathbb{R}^m$ is measurable if E is bounded, the characteristic function χ_E is integrable, and $\chi_{\partial E}$ is integrable and has integral $\int \chi_{\partial E} \,\mathrm{d}\, x = 0$. ♣

Definition 2.5

An exhaustion of a set $E \subset R^m$ is a sequence of measurable subsets E_n such that $E_n \subset E_{n+1} \subset E$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty E_n = E$. ♣

Example 2.2 $E_n := [-n, n]^2$ is a exhaustion of $\mathbb{R}^2$.

Definition 2.6

Let $\{E_n\}$ be an exhaustion of the set E and suppose the function $f : E \to \mathbb{R}$ is integrable on the sets $E_n \in \{E_n\}$. If the limit

$$\int_E f(x)\mathrm{d}x := \lim_{n\to\infty} \int_{E_n} f(x)\mathrm{d}x$$

exists and has a value independent of the choice of the sets in the exhaustion of E, this limit is called the improper integral of f over E. ♣

[1] For the reference of this section, I copied the chapter 11.6 of the book: Zorich, Vladimir Antonovich, and Octavio Paniagua. Mathematical analysis II. Vol. 220. Berlin: Springer, 2016.

We do need to check the independence for all exhaustions. Consider $f(x) = \sin x$. Then $\int_0^{2n\pi} f(x)\,\mathrm{d}\,x = 0$ but $\int_0^{(2n+1)\pi} f(x)\,\mathrm{d}\,x = 2$. So we may have two integral for $\sin x$ if we consider only one exhaustion, which does not make sense.

For non-negative functions we don't need to check for all exhaustions.

Proposition 2.1

If a function $f : E \to \mathbb{R}$ is nonnegative and the limit in Definition 2.6 exists for even one exhaustion $\{E_n\}$ of the set E, then the improper integral of f over E converges.

Example 2.3 Let us find the improper integral $\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y$.

We shall exhaust the plane $\mathbb{R}^2$ by the sequence of disks $E_n = \left\{(x, y) \in \mathbb{R}^2 \mid\ x^2 + y^2 < n^2\right\}$. After passing to polar coordinates we find easily that

$$\iint_{E_n} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\varphi \int_0^n \mathrm{e}^{-r^2}\,\mathrm{d}r = \pi\left(1 - \mathrm{e}^{-n^2}\right) \to \pi, \quad \text{as } n \to \infty.$$

By Proposition 2.1 we can now conclude that this integral converges and equals π. One can derive a useful corollary from this result if we now consider the exhaustion of the plane by the squares $E'_n = \left\{(x, y) \in \mathbb{R}^2 || x| \leq n \wedge |y| \leq n\right\}$.

$$\iint_{E'_n} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\,\mathrm{d}y = \int_{-n}^{n} \mathrm{d}y \int_{-n}^{n} \mathrm{e}^{-(x^2+y^2)}\mathrm{d}x = \left(\int_{-n}^{n} \mathrm{e}^{-t^2}\,\mathrm{d}t\right)^2$$

By Proposition 2.1 this last quantity must tend to π as $n \to \infty$. Thus, following Euler and Poisson, we find that

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$$

This is the so-called Gaussian integral, often used in statistics and physics.

2.2 Characteristic function

Let's look at another problem from the assignment:

Problem 2.1 Let S be a non-empty set in $\mathbb{R}^n$. Define its characteristic function χ_S to be $\chi_S(\mathbf{x}) = 1$ for $\mathbf{x} \in S$ and $\chi_S(\mathbf{x}) = 0$ otherwise. Prove the following identities: (a) $\chi_{A\cap B} = \chi_A\chi_B$. (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

We can prove for more general cases. Let X, Y be any set. Denote the set of all maps from X to Y by Y^X. Here we consider $Y = \{0, 1\}$. Then the elements in $\{0, 1\}^X$ are called the characteristic functions on X. Any such function is defined in the same way as χ_S for some $S \subset X$.

To prove two functions are equal, it is to prove their values at each point are equal.

We first consider the equation $\chi_{A\cap B} = \chi_A\chi_B$. By definition $\chi_{A\cap B}(x) = 1$ if and only if $x \in A \cap B$. And $\chi_A(x)\chi_B(x) = 1$ if and only if $\chi_A(x) = \chi_B(x) = 1$, if and only if $x \in A$ and $x \in B$, which is equivalent to $x \in A \cap B$. Therefore we have $\chi_{A\cap B} = \chi_A\chi_B$.

For (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ we can argue in a similar way by checking when the both sides equal to 1. There is another method to prove it and I would like to show that for general cases. Firstly we assume the following formula for the indeterminants $a_1, \cdots, a_n$.

Lemma 2.1

$$(1 - a_1)(1 - a_2) \times \cdots \times (1 - a_n) = \sum_{k=1}^{n} \prod_{1\leq i_1<\cdots<i_k\leq n} (-1)^k a_{i_1}a_{i_2}\cdots a_{i_k}$$

Proposition 2.2

Let $S_1, S_2, \cdots, S_n$ be subsets of X, then we have

$$\chi_{S_1\cup\cdots\cup S_n} = \sum_{k=1}^{n} \prod_{1\le i_1<\cdots<i_k\le n} (-1)^{k-1}\chi_{S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}}$$

Proof By substituting S_i into a_i in lemma 2.1, we have

$$(\chi_X - \chi_{S_1})(\chi_X - \chi_{S_2}) \times \cdots \times (\chi_X - \chi_{S_n}) = \sum_{k=1}^{n} \prod_{1\le i_1<\cdots<i_k\le n} (-1)^k \chi_{S_{i_1}}\chi_{S_{i_2}}\cdots\chi_{S_{i_k}}$$

By induction on (a) we know that $\chi_{S_{i_1}}\chi_{S_{i_2}}\cdots\chi_{S_{i_k}} = \chi_{S_{i_1}\cap S_{i_2}\cap\cdots\cap S_{i_k}}$.

Since $\chi_S + \chi_{S^c} = \chi_X$, for the left part we have

$$\begin{aligned}(\chi_X - \chi_{S_1})(\chi_X - \chi_{S_2}) \times \cdots \times (\chi_X - \chi_{S_n}) &= \chi_{S_1^c}\chi_{S_2^c}\cdots\chi_{S_n^c} \\ &= \chi_{S_1^c\cap\cdots\cap S_n^c} \\ &= \chi_{(S_1\cup\cdots\cup S_n)^c} \\ &= \chi_X - \chi_{S_1\cup\cdots\cup S_n}\end{aligned}$$

Substract χ_X from both sides we proved the proposition.

Remark This is a kind of the so-called inclusion exlusion principle.

Remark For more generalizations, investigate the term ”Boolean ring”, ”fuzzy set”.

2.3 Volume of tetrahedron[2]

Now we switch to another problem. We know that the area of a triangle is $\frac{1}{2}a \times h$ where a is the length of the base of the triangle and h is the height to the base. While for a tetrahedron we know its volume is $\frac{1}{3}S \times h$ where S is the area of the base of the tetrahedron and h is the height to the base.

One might imagine how those people living in a world of dimension 4 calculate the ”volume” of a ”tetrahedron” of dimension 4 and one may guess the formula $\frac{1}{4}S \times h$ still holds. And generally, aliens in $\mathbb{R}^n$ should have the formula $\frac{1}{n}S \times h$.

To check this, one could first assume that the formula hold for all ”tetrahedrons” if and only if it hold for a standard ”tetrahedron”. We define the standard ”tetrahedron” in $\mathbb{R}^m$ to be the subset $\Delta_m := \{(x_1, \cdots, x_m) \in \mathbb{R}^m | 0 \le x_i \le 1, \forall 1 \le i \le m, x_1 + \cdots + x_m \le 1\}$, we call Δ_m the standard simplex.

The area of Δ_2 is

$$\int_0^1 \int_0^{1-x_2} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \frac{1}{2}$$

The volume of Δ_3 is

$$\int_0^1 \int_0^{1-x_3} \int_0^{1-x_2-x_3} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 = \frac{1}{2} \times \frac{1}{3}$$

Analogously you will find the volume of Δ_m is

$$\int_0^1 \int_0^{1-x_m} \cdots \int_0^{1-x_3-\cdots-x_m} \int_0^{1-x_2-x_3-\cdots-x_m} 1 \,\mathrm{d}x_1 \,\mathrm{d}x_2 \cdots \mathrm{d}x_m = \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{m} = \frac{1}{m!}$$

So we do find that the formula $\frac{1}{m}S \times h$ hold for the volume of ”tetrahedron” in dimension m.

Leave it behind, now consider the following experiment: Let’s pick elements from $[0, 1]$ randomly, until

[2]The reference for this section is https://zhuanlan.zhihu.com/p/369714158

their sum gets larger than 1. Then we record the number of elements we have chosen. For example, if we got two random numbers 0.12, 0.57 at first and we got 0.41 for the third pick, then we record $n_1 = 3$ as now $0.12 + 0.57 + 0.41 > 1$. Now we do this step repeatedly and we get a sequence of number $n_1, n_2, \cdots, n_k, \cdots$. We call $A = \lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} n_i$ the average of number of tries that we need to pick the numbers until their sum is larger than 1. Now we try to calculate it.

Let $\Pr[Y = i]$ be the probability that we need to pick exactly i numbers from $[0, 1]$ so that their sum exceeds 1. A probability theorem will tell you that

$$A = \sum_{i=1}^{\infty} i \cdot \Pr[Y = i]$$

We can rewrite it in the following way

$$\begin{aligned}
\sum_{i=1}^{\infty} i \cdot \Pr[Y = i] &= 1 \cdot \Pr[Y = 1] + 2 \cdot \Pr[Y = 2] + 3 \cdot \Pr[Y = 3] + \dots \\
&= \sum_{i=1}^{\infty} \Pr[Y = i] + (1 \cdot \Pr[Y = 2] + 2 \cdot \Pr[Y = 3] + \dots) \\
&= \sum_{i=1}^{\infty} \Pr[Y = i] + \sum_{i=2}^{\infty} \Pr[Y = i] + (1 \cdot \Pr[Y = 3] + 2 \cdot \Pr[Y = 4] + \dots) \\
&= \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \Pr[Y = i] \\
&= \sum_{k=1}^{\infty} \Pr[Y \geq i]
\end{aligned}$$

If we choose i times from $[0, 1]$, then we have the possible numbers $a_1, a_2, \cdots, a_i$. Then $\Pr[Y \geq i]$ is the probability that $a_1 + a_2 + \cdots + a_{i-1} \leq 1$, so it is exactly the quotient of the volume of Δ_{i-1} by the volume of $[0, 1]^{i-1}$, which is $\frac{1}{(i-1)!}$. Therefore, the avarage of number of tries $A = \sum_{i=1}^{\infty} \frac{1}{(i-1)!} = e$.

Remark The above procedure is a way to approximate e using Monte Carlo Simulation.

Exercise 2.1 Try to find the average of number of tries that we need to pick numbers from $[0, 1]$ randomly until their sum exceeds $x \in [0, 1]$.

Tutorial 3 2022/10/12

3.1 Average

When we calculate the average of the scores, we sum the scores of every students and calculate the quotient of the sum by the number of students. If we see that the integration is a kind of "sum" in a generalized sense, then the average we met in this course is nothing different as the the original ones that we met in high schools.

It can also have another explanation. Let $D \subset \mathbb{R}^2$ be a domain and f be an integrable function over D. Let's choose n points in D randomly, say they are $x_1, x_2, \cdots, x_n$. Then we calculate the average value $\overline{f}$ of f over $x_1, x_2, \cdots, x_n$ as follows

$$\overline{f} = \frac{\sum_{i=1}^{n} f(x_i)}{n}$$

When n is large enough, we would hope the average will approach the average of f over D, $\frac{1}{|D|}\iint_D f dA$. That is the following equality holds

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{|D|}\iint_D f dA.$$

There is still something that is not clear enough, what do we mean by choosing points in D "randomly"? It can mean any points have the equal probability to be chosen from D. To be more precise, let R be any subdomain of D, then we require the infinite sequence $\{x_i\}_{i\in\mathbb{N}}$ we have chosen should satisfy

$$\lim_{n\to\infty} \frac{|\{x_1, \ldots, x_n\} \cap R|}{n} = \frac{|R|}{|D|}.$$

We say the sequence $\{x_i\}_{i\in\mathbb{N}}$ that satisfies the above equation for any $R \subset D$ is **equidistributed** on D.

In fact we have

Theorem 3.1

A sequence of points $(x_1, x_2, \cdots)$ in D is equidistributed if and only if for every integrable function f we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} f(x_i)}{n} = \frac{1}{|D|}\iint_D f dA.$$

♡

Besides discrete points, we could also consider the curves running in the domain D "randomly" or "equidistributedly". Let $c : \mathbb{R}_{\geq 0} \to D \subset \mathbb{R}^2$ be such a curve. Then we can also make the following equality make sense

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(c(t))dt = \frac{1}{|D|}\iint_D f dA.$$

In the so called Ergodic theory, we call the left hand side of the equality the **time average** of f and the right hand side the **space average** of f.

3.2 Symmetry

The above discussion shows a way to calculate the integral of f over a domain D. Which would also be pretty consistent with our intuition.

Let's look at an example, the supplementary problems 3 in the exercise sheet 4:

Problem 3.1 Let D be a region in the plane which is symmetric with respect to the origin, that is, $(x, y) \in D$ if and only if $(-x, -y) \in D$. Show that

$$\iint_D f(x, y) dA(x, y) = 0,$$

when f is odd, that is, $f(-x, -y) = -f(x, y)$ in D. Suggestion: Use polar coordinates.

Proof We give two proof, one based on the hint and one based on theorem 3.1.

1. Solution. Let $\tilde{f}$ be the universal extension of f. It is readily checked that $\tilde{f}$ is an odd function in the entire plane. Let D_1 be a large disk of radius R centered at the origin containing D. By converting to polar coordinates,

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f} dA \\ &= \int_0^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta + \int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta.\end{aligned}$$

Further, using the change of variables $\alpha = \theta - \pi$, the second integral becomes

$$\begin{aligned}\int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos(\alpha+\pi), r\sin(\alpha+\pi)) r dr d\alpha \\ &= \int_0^{\pi} \int_0^R \tilde{f}(-r\cos\alpha, -r\sin\alpha) r dr d\alpha \\ &= -\int_0^{\pi} \int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r dr d\alpha\end{aligned}$$

It follows that

$$\begin{aligned}\iint_D f &= \iint_{D_1} \tilde{f} dA \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta + \int_{\pi}^{2\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta \\ &= \int_0^{\pi} \int_0^R \tilde{f}(r\cos\theta, r\sin\theta) r dr d\theta - \int_0^{\pi} \int_0^R \tilde{f}(r\cos\alpha, r\sin\alpha) r dr d\alpha = 0.\end{aligned}$$

2. Choose a equidistributed sequence $(p_1, p_2, \cdots)$ of points in D, where $p_i = (x_i, y_i) \in D \subset \mathbb{R}^2$. Let $q_i = -p_i = (-x_i, -y_i)$, then the sequence of points $(p_1, q_1, p_2, q_2, \cdots)$ is still equidistributed. By theorem 3.1, we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n f(p_i) + \sum_{i=1}^n f(q_i)}{2n} = \frac{1}{|D|} \iint_D f dA.$$

We have $f(p_i) + f(q_i) = f(p_i) + f(-p_i) = f(p_i) - f(p_i) = 0$, so the right hand side equal to 0. (You may not use this method in the exams unless you write down the proof of theorem 3.1. You may find the proof by searching the term "equidistributed sequence".)

You can prove similar results in the similar manner.

Problem 3.2 Let D be a region in the plane which is symmetric with respect to the x-axis, that is, $(x, y) \in D$ if and only if $(x, -y) \in D$. Show that

$$\iint_D f(x, y) dA(x, y) = 0,$$

when f is odd in the y-direction, that is, $f(x, -y) = -f(x, y)$ in D.

Problem 3.3 Let T be map on $\mathbb{R}^2$ which will rotate $\mathbb{R}^2$ by $\frac{2\pi}{3}$. So $T(x,y) = \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)$.

Let D be a region in the plane that $(x,y) \in D$ if and only if $T(x,y) \in D$.

Let f_1 and f_2 be two functions such that

$$f_1(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y) = -\frac{1}{2}f_1(x,y) - \frac{\sqrt{3}}{2}f_2(x,y)$$
$$f_2(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y) = \frac{\sqrt{3}}{2}f_1(x,y) - \frac{1}{2}f_2(x,y)$$

Show that

$$\iint_D f_1(x,y)dA(x,y) = \iint_D f_2(x,y)dA(x,y) = 0,$$

3.3 Mean Motion[1]

Conversely, theorem 3.1 can also be used to calculate some kind of limits which are hard to calculate through routine methods, by doing multiple integration.

Let us now turn our attention to the universe. Let's assume the sun, the earth and the moon are lying in the same plane. After we know the angular velocity of the earth rotating around sun and the angular velocity of the moon rotating around the earth, it is natural to ask what is the average angular velocity of the moon rotating around the sun.

The mathematics model for this problem is as follows. Let a_1, a_2 be two positive numbers represent the distance between the sun and the earth and the moon and the earth. Let ω_1 and ω_2 be the corresponding angle velocities. Then the average angular velocity of the moon around the sun is

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t})$$

Here i denotes the complex number $\sqrt{-1}$ and Arg denotes the variation of the total angle of the particle $a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}$ since time $t = 0$.

When $a_1 = a_2 = a$, we can calculate the limit directly.

$$\begin{aligned}
& ae^{i\omega_1 t} + ae^{i\omega_2 t} \\
&= a\left(\cos\omega_1 t + \cos\omega_2 t\right) + i\left(\sin\omega_1 t + \sin\omega_2 t\right) \\
&= 2a\left(\cos\frac{\omega_1+\omega_2}{2}t\cos\frac{w_1-w_2}{2}t + i\sin\frac{w_1+w_2}{2}t\cos\frac{w_1-w_2}{2}t\right) \\
&= 2a\cos^{\frac{w_1-w_2}{2}t} e^{\frac{w_1+w_2}{2}t}
\end{aligned}$$

So

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(e^{\frac{w_1+w_2}{2}t}) = \frac{w_1+w_2}{2}$$

The average angular velocity is the average of ω_1 and ω_2.

For $a_1 \neq a_2$, things will become much more difficult. A brilliant method Weyl used in his paper *Mean Motion* is to turn it into the calculation of space average.

[1]The reference for this section is Herman Weyl's paper *Mean Motion*, published in 1938, American Journal of Mathematics

Let f be a function on $\mathbb{R}^2$ defined as follows

$$f\left(\theta_1, \theta_2\right) = \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right)$$

Since a modification of the starting angle will not affect the average velocity, we have

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)})$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \lim_{t\to\infty} \frac{1}{t} \int_0^t f(\theta_1 + \omega_1 \tau, \theta_2 + \omega_2 \tau) d\tau$$

According to the definition f is period such that $f(x, y) = f(x + 2\pi, y) = f(x, y + 2\pi)$, so we may view $(\omega_1 \tau, \omega_2 \tau)$ as a curve in $[0, 2\pi]^2$. As the time average equals the space average, we have

$$\frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f\left(\theta_1, \theta_2\right) d\theta_1 d\theta_2 = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\theta_1 + \omega_1 t, \theta_2 + \omega_2 t\right) dt$$

Combine all of the above and we turned the calculation of limit into the calculation of the integral

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) d\theta_1 d\theta_2$$

Since

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) = \left.\frac{d}{dt}\right|_{t=0} \tan^{-1}\left(\frac{a_1 \sin\left(\theta_1 + w_1 t\right) + a_2 \sin\left(\theta_2 + w_2 t\right)}{a_1 \cos\left(\theta_1 + w_1 t\right) + a_2 \cos\left(\theta_2 + w_2 t\right)}\right)$$

by estimatedly calculation we can assume

$$\left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + w_1 t)} + a_2 e^{i(\theta_2 + w_2 t)}\right) = p_1(\theta_1, \theta_2)\omega_1 + p_2(\theta_1, \theta_2)\omega_2$$

Let

$$\overline{p_i} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} p_i(\theta_1, \theta_2) d\theta_1 d\theta_2$$

Then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t}) = \omega_1 \overline{p_1} + \omega_2 \overline{p_2}$$

To calculate $\overline{p_1}$, let $\omega_1 = 1, \omega_2 = 0$, then

$$\begin{aligned}
\overline{p_1} &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \left.\frac{d}{dt}\right|_{t=0} \operatorname{Arg}\left(a_1 e^{i(\theta_1 + t)} + a_2 e^{i\theta_2}\right) d\theta_1 d\theta_2 \\
&= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d}{d\theta_1} \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right) d\theta_1 d\theta_2
\end{aligned}$$

And

$$\begin{aligned}
&\int_0^{2\pi} \frac{d}{d\theta_1} \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right) d\theta_1 \\
&= \operatorname{Arg}\left(a_1 e^{i\theta_1} + a_2 e^{i\theta_2}\right)\Big|_{a_1=0}^{\theta_1=2\pi} \\
&= \begin{cases} 2\pi, & a_1 > a_2 \\ 0, & a_1 < a_2 \end{cases}
\end{aligned}$$

Therefore we have $\overline{p_1} = 1, \overline{p_2} = 0$ if $a_1 > a_2$ or $\overline{p_2} = 1, \overline{p_1} = 0$ if $a_2 > a_1$.

To conclude, we see that the average angular velocity only depend on angular velocity of the particle with longer radius. So we proved that the average angular velocity of the moon rotating about the sun is the same as the angular velocity of the earth.

Remark The above argument requires that $\frac{\omega_1}{\omega_2}$ is not a rational number in order to guarentee that the path $(\omega_1 t, \omega_2 t) \mod 2\pi$ is equidistributed on $[0, 2\pi]^2$. If $\frac{\omega_1}{\omega_2}$ is rational, the result still holds as the limit should be continuous on (ω_1, ω_2).

✍ **Exercise 3.1** Calculate

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t})$$

Show that if a_1, a_2, a_3 form a triangle whose corresponding interior angles are $\alpha_1, \alpha_2, \alpha_3$, then

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t}) = \frac{\alpha_1}{\pi}\omega_1 + \frac{\alpha_2}{\pi}\omega_2 + \frac{\alpha_3}{\pi}\omega_3$$

✍ **Exercise 3.2** Say something about

$$\lim_{t\to\infty} \frac{1}{t} \operatorname{Arg}(a_1 e^{i\omega_1 t} + a_2 e^{i\omega_2 t} + a_3 e^{i\omega_3 t} + a_4 e^{i\omega_4 t})$$

✍ **Exercise 3.3** Prove that the mean of the square of the length $\lim_{t\to\infty} \frac{1}{T} \int_0^T |\sum_{k=1}^N a_k e^{i\omega_k t}|^2 dt$ is $\sum_{k=1}^N |a_k|^2$. (Hint: By theorem 3.1 we have $\lim_{t\to\infty} \frac{1}{T} \int_0^T |\sum_{k=1}^N a_k e^{i\omega_k t}|^2 dt = \frac{1}{(2\pi)^N} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} |\sum_{k=1}^N a_k e^{i\theta_k}|^2 d\theta_1 d\theta_2 \cdots d\theta_N$ and $|\sum_{k=1}^N a_k e^{i\theta_k}|^2 = \sum_{k=1}^N |a_k|^2 + \sum_{j=1}^N \sum_{l=1}^N a_j a_l e^{i(\theta_j - \theta_l)}$)

Tutorial 4 2022.10.19

Let $a_1, a_2, \cdots, a_n$ be n positive numbers.

We define

$$D(a_1, a_2, \cdots, a_n) := \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1^{a_1} + \cdots + x_n^{a_n} \leq 1, \text{ and } x_i \geq 0, \forall i = 1, 2, \cdots, n\}.$$

Let's call $D(a_1, a_2, \cdots, a_n)$ a **spherical type domain** (named by me temporarily, since I don't know whether it has an official name). We are going to calculate the volume of $D(a_1, a_2, \cdots, a_n)$, denoted by $V(a_1, a_2, \cdots, a_n)$.

For example, $V(2,2)$ is a quarter of the volume of the disk $\mathbb{B}^2 = \{(x,y)|x^2+y^2 \leq 1\}$ and $V(2,2,2)$ is one-eighth of the volume of the ball $\mathbb{B}^3 = \{(x,y,z)|x^2+y^2+z^2 \leq 1\}$. That's why we call it spherical type.

You can obtain the volume of $\mathbb{B}^2$ and $\mathbb{B}^3$ immediately, but we will present here a unified approach to calculate $V(a_1, a_2, \cdots, a_n)$. To begin with, we need some extra ingredients.

4.1 Gamma function

The **Gamma function** Γ is a function defined over positive real numbers $\alpha \in \mathbb{R}_{>0}$.

$$\Gamma(\alpha) := \int_0^{+\infty} x^{\alpha-1} \mathrm{e}^{-x} \, \mathrm{d}x$$

We have obviously

$$\Gamma(1) = \int_0^{\infty} e^{-t} dt = 1$$

and for $x > 0$, an integration by parts yields

$$\Gamma(x+1) = \int_0^{\infty} t^x e^{-t} dt = \left[-t^x e^{-t}\right]_0^{\infty} + x \int_0^{\infty} t^{x-1} e^{-t} dt = x\Gamma(x),$$

and the relation

$$\Gamma(x+1) = x\Gamma(x) \tag{4.1}$$

is the important functional equation. For integer values the functional equation becomes

$$\Gamma(n+1) = n!,$$

and it's why the gamma function can be seen as an extension of the factorial function to real non null positive numbers.

The values of Gamma function at half integers can also be calculated:

The change of variable $t = u^2$ gives

$$\Gamma(1/2) = \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt = 2 \int_0^{\infty} e^{-u^2} du = 2\frac{\sqrt{\pi}}{2} = \sqrt{\pi}.$$

This is the Gaussian integral that we have met in assignment 3. The functional equation 4.1 entails for positive integers n

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{1.3.5 \ldots (2n-1)}{2^n} \sqrt{\pi}, \tag{4.2}$$

We also have

Proposition 4.1

$$\frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

Proof Let $x = \frac{y}{1+y}$, we have $\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \int_0^{+\infty} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}dy$. So

$$\begin{aligned}
\Gamma(\alpha+\beta)\cdot\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx &= \int_0^{+\infty}\frac{\Gamma(\alpha+\beta)y^{\alpha-1}}{(1+y)^{\alpha+\beta}}\mathrm{d}y = \\
&= \int_0^{+\infty}\left(y^{\alpha-1}\int_0^{+\infty} x^{\alpha+\beta-1}\mathrm{e}^{-(1+y)x}\,\mathrm{d}x\right)\mathrm{d}y \overset{!}{=} \\
&\overset{!}{=} \int_0^{+\infty}\left(\int_0^{+\infty} y^{\alpha-1}x^{\alpha+\beta-1}\mathrm{e}^{-(1+y)x}\,\mathrm{d}y\right)\mathrm{d}x = \\
&= \int_0^{+\infty}\left(x^{\beta-1}\mathrm{e}^{-x}\int_0^{+\infty}(xy)^{\alpha-1}\mathrm{e}^{-xy}x\,\mathrm{d}y\right)\mathrm{d}x = \\
&= \int_0^{+\infty}\left(x^{\beta-1}\mathrm{e}^{-x}\int_0^{+\infty}u^{\alpha-1}\mathrm{e}^{-u}\,\mathrm{d}u\right)\mathrm{d}x = \Gamma(\alpha)\cdot\Gamma(\beta).
\end{aligned}$$

4.2 Dirichlet's integral[1]

We shall now show how the repeated integral

$$I = \iint\ldots\int f\left(t_1+t_2+\ldots+t_n\right)t_1^{\alpha_1-1}t_2^{\alpha_2-1}\ldots t_n^{\alpha_n-1}dt_1dt_2\ldots dt_n$$

may be reduced to a simple integral, where f is continuous, $\alpha_r > 0(r = 1,2,\ldots n)$ and the integration is extended over all positive values of the variables such that $t_1+t_2+\ldots+t_n \leqslant 1$. To simplify $\int_0^{1-\lambda}\int_0^{1-\lambda-T} f(t+T+\lambda)t^{\alpha-1}T^{\beta-1}dtdT$ (where we have written t, T, α, β for $t_1, t_2, \alpha_1, \alpha_2$ and λ for $t_3+t_4+\ldots+t_n$), put $t = T(1-v)/v$; the integral becomes (if $\lambda \neq 0$)

$$\int_0^{1-\lambda}\int_{T/(1-\lambda)}^1 f(\lambda+T/v)(1-v)^{\alpha-1}v^{-\alpha-1}T^{\alpha+\beta-1}dvdT.$$

Changing the order of integration, the integral becomes

$$\int_0^1\int_0^{(1-\lambda)v} f(\lambda+T/v)(1-v)^{a-1}v^{-a-1}T^{a+\beta-1}dTdv.$$

Putting $T = v\tau_2$ and by proposition 4.1, the integral becomes

$$\int_0^1\int_0^{1-\lambda} f\left(\lambda+\tau_2\right)(1-v)^{\alpha-1}v^{\beta-1}\tau_2^{\alpha+\beta-1}d\tau_2dv = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^{1-\lambda} f\left(\lambda+\tau_2\right){\tau_2}^{\alpha+\beta-1}d\tau_2.$$

Hence

$$I = \frac{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)}{\Gamma\left(\alpha_1+\alpha_2\right)}\iint\ldots\int f\left(\tau_2+t_3+\ldots+t_n\right)\tau_2^{a_1+a_2-1}t_3^{a_3-1}\ldots t_n^{a_n-1}d\tau_2dt_3\ldots dt_n,$$

the integration being extended over all positive values of the variables such that $\tau_2+t_3+\ldots+t_n \leqslant 1$.

Continually reducing in this way we get

$$I = \frac{\Gamma\left(\alpha_1\right)\Gamma\left(\alpha_2\right)\ldots\Gamma\left(\alpha_n\right)}{\Gamma\left(\alpha_1+\alpha_2+\ldots+\alpha_n\right)}\int_0^1 f(\tau)\tau^{\Sigma_{i=1}^n\alpha_i-1}d\tau,$$

which is Dirichlet's result, formally

[1] The reference for this section is page 258, Whittaker, Edmund T., and George Neville Watson. A course of modern analysis: an introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. University press, 1920.

Theorem 4.1

$$\int_0^1\int_0^{1-t_n}\cdots\int_0^{1-t_3-t_4-\cdots-t_n}\int_0^{1-t_2-t_3-\cdots-t_n} f\left(\sum_{i=1}^n t_i\right) t_1^{\alpha_1-1}t_2^{\alpha_2-1}\ldots t_n^{\alpha_n-1}dt_1dt_2\ldots dt_{n-1}dt_n =$$
$$=\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}\int_0^1 f(\tau)\tau^{\Sigma_{i=1}^n\alpha_i-1}d\tau$$

4.3 Volume of spherical type domain

Recall that the domain is defined as

$$D(a_1,a_2,\cdots,a_n):=\{(x_1,\cdots,x_n)\in\mathbb{R}^n|x_1^{a_1}+\cdots+x_n^{a_n}\le 1,\text{ and } x_i\ge 0,\forall i=1,2,\cdots,n\}.$$

So its volume is

$$\int_{D(a_1,a_2,\cdots,a_n)} 1dV =$$
$$=\int_0^1\int_0^{(1-x_n^{a_n})^{\frac{1}{a_{n-1}}}}\cdots\int_0^{(1-x_3^{a_3}-x_4^{a_4}-\cdots-x_n^{a_n})^{\frac{1}{a_2}}}\int_0^{(1-x_2^{a_2}-x_3^{a_3}-\cdots-x_n^{a_n})^{\frac{1}{a_1}}} 1dx_1dx_2\ldots dx_{n-1}dx_n \tag{4.3}$$

Let $t_i=x_i^{a_i}, i=1,\cdots,n$. By change of variables fomula, we have,

$$(4.3)=\frac{1}{a_1a_2\cdots a_n}\int_0^1\int_0^{1-t_n}\cdots\int_0^{1-t_3-t_4-\cdots-t_n}\int_0^{1-t_2-t_3-\cdots-t_n} t_1^{\frac{1}{a_1}-1}t_2^{\frac{1}{a_2}-1}\ldots t_n^{\frac{1}{a_n}-1}dt_1dt_2\ldots dt_{n-1}dt_n$$

This is the special case of theorem 4.1 for $f=1,\alpha_i=\frac{1}{a_i}$, so

$$\begin{aligned}(4.3)&=\frac{1}{a_1a_2\cdots a_n}\frac{\Gamma\left(\frac{1}{a_1}\right)\Gamma\left(\frac{1}{a_2}\right)\ldots\Gamma\left(\frac{1}{a_n}\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)}\int_0^1\tau^{\Sigma_{i=1}^n\frac{1}{a_i}-1}d\tau\\
&=\frac{1}{a_1a_2\cdots a_n}\frac{\Gamma\left(\frac{1}{a_1}\right)\Gamma\left(\frac{1}{a_2}\right)\ldots\Gamma\left(\frac{1}{a_n}\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)}\frac{1}{\Sigma_{i=1}^n\frac{1}{a_i}}\\
&=\frac{\Gamma\left(\frac{1}{a_1}+1\right)\Gamma\left(\frac{1}{a_2}+1\right)\ldots\Gamma\left(\frac{1}{a_n}+1\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}+1\right)}\end{aligned}$$

So we get

Theorem 4.2

$$V(a_1,a_2,\cdots,a_n)=\frac{\Gamma\left(\frac{1}{a_1}+1\right)\Gamma\left(\frac{1}{a_2}+1\right)\ldots\Gamma\left(\frac{1}{a_n}+1\right)}{\Gamma\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}+1\right)}$$

The relative advantage of using Gamma function to denote the volume is that we have the Gamma function calculator, for example https://keisan.casio.com/exec/system/1180573444. So we are able to calculate by hands approximately the volume of any spherical type domain $D(a_1,a_2,\cdots,a_n)$ for $a_i>0$.

And for some particular a_i we could obtain the exact value easily.

Example 4.1 For positive integers m,n,l, the volume of the domain $D(m,n,l)$ is $\frac{\Gamma(m+1)\Gamma(n+1)\Gamma(n+1)}{\Gamma(m+n+l+1)}=\frac{m!n!l!}{(m+n+l)!}$

Example 4.2 Consider the domain D defined by $\{(x,y)|x^{\frac{2}{3}}+y^{\frac{2}{3}}\le 1\}$. Then its area is $4V(\frac{2}{3},\frac{2}{3})=$

$\frac{\Gamma(\frac{3}{2}+1)\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{3}{2}+\frac{3}{2}+1)} = \frac{(\frac{3}{4}\sqrt{\pi})^2}{3!} = \frac{3}{32}\pi$ by (4.2). Also you can find the volume of the domain defined by $\{(x,y,z)|x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} \leq 1\}$. In fact for positive integers m, n, l, the volume of $\{(x,y,z)|x^{\frac{2}{m}} + y^{\frac{2}{n}} + z^{\frac{2}{l}} \leq 1\}$ is handleable using theorem 4.2.

Tutorial 5 2022.10.26

5.1 Irrationality of $\zeta(3)$[1]

We give the proof for the irrationality of $\zeta(3)$. This proof is achieved by means of double and triple integrals, the shape of which is motivated by Apéry's formulas. Like Apéry's proof it also works for $\zeta(2)$, which is of course already known to be transcendental since it equals $\pi^2/6$. Most of the integrals that appear in the proof are improper. The manipulations with these integrals can be justified if one replaces $\int_0^1$ by $\int_\varepsilon^{1-\varepsilon}$ and by letting ε tend to zero.

Throughout this paper we denote the lowest common multiple of $1, 2, \ldots, n$ by d_n. The value of d_n can be estimated by

$$d_n = \prod_{\substack{\text{Prime} \\ p \leqslant n}} p^{[\log n/\log p]} < \prod_{\substack{\text{Prime} \\ p \leqslant n}} p^{\log n/\log p},$$

and the latter number is smaller than 3^n for sufficiently large n.

Lemma 5.1

Let r and s be non-negative integers. If $r > s$ then,

(a) $\int_0^1 \int_0^1 \frac{x^r y^s}{1-xy} dxdy$ *is a rational number whose denominator is a divisor of* d_r^2.

(b) $\int_0^1 \int_0^1 -\frac{\log xy}{1-xy} x^r y^s dxdy$ *is a rational number whose denominator is a divisor of* ${d_r}^3$.

If $r = s$, then

(c) $\int_0^1 \int_0^1 \frac{x^r y^r}{1-xy} dxdy = \zeta(2) - \frac{1}{1^2} - \ldots - \frac{1}{r^2}$,

(d) $\int_0^1 \int_0^1 -\frac{\log xy}{1-xy} x^r y^r dxdy = 2\left\{\zeta(3) - \frac{1}{1^3} - \ldots - \frac{1}{r^3}\right\}$.

♡

Proof Let σ be any non-negative number. Consider the integral

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} dxdy \tag{5.1}$$

Develop $(1-xy)^{-1}$ into a geometrical series and perform the double integration. Then we obtain

$$\sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} \tag{5.2}$$

Assume that $r > s$. Then we can write this sum as

$$\sum_{k=0}^{\infty} \frac{1}{r-s}\left\{\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1}\right\} = \frac{1}{r-s}\left\{\frac{1}{s+1+\sigma} + \ldots + \frac{1}{r+\sigma}\right\}. \tag{5.3}$$

If we put $\sigma = 0$ then assertion (a) follows immediately. If we diflerentiate with respect to σ and put $\sigma = 0$, then integral 5.1 changes into

$$\int_0^1 \int_0^1 \frac{\log xy}{1-xy} x^r y^s dxdy$$

and summation 5.3 becomes

$$\frac{-1}{r-s}\left\{\frac{1}{(s+1)^2} + \ldots + \frac{1}{r^2}\right\}.$$

---

[1]This is a copy of Beukers, Frits. "A note on the irrationality of $\zeta(2)$ and $\zeta(3)$." Pi: A Source Book. Springer, New York, NY, 2004. 434-438.

Assertion (b) now follows straight away. Assume $r = s$, then by 5.1 and 5.2,

$$\int_0^1 \int_0^1 \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} dxdy = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}.$$

By putting $\sigma = 0$ assertion (c) becomes obvious. Differentiate with respect to σ and put $\sigma = 0$. Then we obtain

$$\int_0^1 \int_0^1 \frac{\log xy}{1-xy} x^r y^r dxdy = \sum_{k=0}^{\infty} \frac{-2}{(k+r+1)^3},$$

which proves assertion (d).

Theorem 5.1

$\zeta(3)$ is irrational.

Proof

Consider the integral

$$\int_0^1 \int_0^1 \frac{-\log xy}{1-xy} P_n(x) P_n(y) dxdy, \tag{5.4}$$

where $n! P_n(x) = \left\{\frac{d}{dx}\right\}^n x^n(1-x)^n$. It is clear from Lemma 5.1 that integral 5.4 equals $(A_n + B_n\zeta(3))\, d_n^{-3}$ for some $A_n \in \mathbb{Z}$, $B_n \in \mathbb{Z}$. By noticing that

$$\frac{-\log xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z} dz$$

integral (6) can be written as

$$\int \frac{P_n(x)P_n(y)}{1-(1-xy)z} dxdydz$$

where $\int$ denotes the triple integration. After an n-fold partial integration with respect to x our integral changes into

$$\int \frac{(xyz)^n (1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}} dxdydz \tag{5.5}$$

Substitute

$$w = \frac{1-z}{1-(1-xy)z}.$$

We obtain

$$\int (1-x)^n (1-w)^n \frac{P_n(y)}{1-(1-xy)w} dxdydw.$$

After an n-fold partial integration with respect to y we obtain

$$\int \frac{x^n(1-x)^n y^n (1-y)^n w^n (1-w)^n}{(1-(1-xy)w)^{n+1}} dxdydw.$$

It is straightforward to verify that the maximum of

$$x(1-x)y(1-y)w(1-w)(1-(1-xy)w)^{-1}$$

occurs for $x = y$ and then that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4 \text{ for all } 0 \leqslant x, y, w \leqslant 1.$$

Hence integral 5.4 is bounded above by

$$(\sqrt{2}-1)^{4n} \int \frac{1}{1-(1-xy)w} dxdydw = (\sqrt{2}-1)^{4n} \int_0^1 \int_0^1 \frac{-\log xy}{1-xy} dxdy = 2(\sqrt{2}-1)^{4n}\zeta(3).$$

Since integral 5.5 is not zero we have

$$0 < |A_n + B_n\zeta(3)|\, d_n^{-3} < 2\zeta(3)(\sqrt{2}-1)^{4n}$$

and hence

$$0 < |A_n + B_n\zeta(3)| < 2\zeta(3)d_n^3(\sqrt{2}-1)^{4n} < 2\zeta(3)27^n(\sqrt{2}-1)^{4n} < \left(\frac{4}{5}\right)^n$$

for sufficiently large n, which implies the irrationality of $\zeta(3)$.

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6.1 Midterm question 8

Problem 6.1

1. Let f and g be continuous on the region D in $\mathbb{R}^3$. Prove the inequality
$$2\iiint_D |fg|dV \leq \alpha^2 \iiint_D f^2 dV + \frac{1}{\alpha^2}\iiint_D g^2 dV,$$
where α is a positive number. Hint: Use $(a \pm b)^2 \geq 0$.
2. Prove
$$\iiint_D |fg|dV \leq \sqrt{\iiint_D f^2 dV}\sqrt{\iiint_D g^2 dV}$$
Hint: Make a good choice of α in the first inequality of 1..

Proof Assume 1. let's prove 2.. If $\iiint_D f^2 dV = 0$, then $f = 0$ on D, so $\iiint_D |fg|dV = 0$ and the inequality holds.

Therefore we may assume $\iiint_D f^2 dV \neq 0$ and $\iiint_D g^2 dV \neq 0$.

Consider the function $f(\alpha) = u\alpha^2 + v\frac{1}{\alpha^2}, \alpha > 0$ where $u > 0, v > 0$, then $\frac{d}{d\alpha} f(\alpha) = 2u\alpha - 2v\frac{1}{\alpha^3}$.

Let $\frac{d}{d\alpha} f(\alpha) = 2u\alpha - 2v\frac{1}{\alpha^3} = 0$, we get when $\alpha^2 = \sqrt{\frac{v}{u}}$ the function f takes its minimal value $2\sqrt{uv}$.

In our case, let $u = \iiint_D f^2 dV$ and $v = \iiint_D g^2 dV$, then $f(\alpha) \geq 2\iiint_D |fg|dV$ implies
$$f((\frac{v}{u})^{\frac{1}{4}}) = 2\sqrt{uv} = 2\sqrt{\iiint_D f^2 dV \iiint_D g^2 dV} \geq 2\iiint_D |fg|dV$$

6.2 Elliptic integral

6.2.1 Perimeter of an eclipse

So how do you understand π, the ratio of a circle's perimeter to its diameter? If you take it for granted, then you could use it to do a lot of things and got many other calculation involving it.

Now think about the perimeter of an eclipse. Do it have the same property as the mysterious number π?

Consider an ellipse with major and minor arcs $2a$ and $2b$ and eccentricity $e := \left(a^2 - b^2\right)/a^2 \in [0, 1)$, e.g.,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
What is the arclength $\ell(a; b)$ of the ellipse, as a function of a and b? Let $x = a\cos\theta, y = b\sin\theta, \quad a > b > 0$, then it has length
$$\begin{aligned}\ell(a; b) &= 4\int_0^{\pi/2} \sqrt{dx^2 + dy^2} = 4\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta \\ &= 4\int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2)\sin^2\theta}d\theta = 4a\int_0^{\pi/2} \sqrt{1 - e^2\sin^2\theta}d\theta,\end{aligned}$$
Let $z = \sin\theta$, then
$$\begin{aligned}\ell(a, b) &= 4a\int_0^1 \sqrt{\frac{1 - ez^2}{1 - z^2}}dz \\ &= 4a\int_0^1 \frac{1 - ez^2}{\sqrt{(1 - ez^2)(1 - z^2)}}dz.\end{aligned}$$

We can not find the exact value of $\ell(a,b)$ by hands since the functions $\sqrt{1-e^2\sin^2\theta}$ and $\frac{1-ez^2}{\sqrt{(1-ez^2)(1-z^2)}}$ can be proved to have no elementary anti-derivatives. However, we could assume we know its value just like how we treat π.

6.2.2 Three kinds of elliptic integral

So here is a generalization. An **elliptic integral** is any integral of the general form

$$f(x)=\int \frac{A(x)+B(x)}{C(x)+D(x)\sqrt{S(x)}}dx$$

where $\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x}), \boldsymbol{C}(\boldsymbol{x})$ and $\boldsymbol{D}(\boldsymbol{x})$ are polynomials in $\boldsymbol{x}$ and $\boldsymbol{S}(\boldsymbol{x})$ is a polynomial of degree 3 or 4.

For some cases they are named

Definition 6.1

1. *The* ***incomplete elliptic integral of the first kind*** *is defined as*

$$u=F(k,\phi)=\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\quad 0<k<1,$$

where ϕ is the amplitude of $F(k,\phi)$ or u, written $\phi=am\ u$, and k is the modulus, $k=\operatorname{mod}u$. The integral is also called Legendre 's form for the elliptic integral of the first kind. If $\phi=\pi/2$, the integral is called the ***complete integral of the first kind***, *denoted by $K(k)$, or simply K.*

2. *The* ***incomplete elliptic integral of the second kind*** *is defined by*

$$E(k,\phi)=\int_0^\phi \sqrt{1-k^2\sin^2\theta}d\theta,\quad 0<k<1,$$

also called Legendre's form for the elliptic integral of the second kind. If $\phi=\pi/2$, the integral is called the ***complete elliptic integral of the second kind***, *denoted by $E(k)$, or simply E. This is the form that arises in the determination of the length of arc of an ellipse. For example, $\ell(a,b)=4aE(e)$.*

3. *The* ***incomplete elliptic integral of the third kind*** *is defined by*

$$H(k,n,\phi)=\int_0^\phi \frac{d\theta}{\left(1+n\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}},\quad 0<k<1, n\neq 0,$$

also called Legendre's form for the elliptic integral of the third kind.

♣

Remark If the transformation $v=\sin\theta$ is made in the Legendre forms, we obtain the following integrals, with $x=\sin\phi$

$$\begin{aligned}
F_1(k,x)&=\int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}\\
E_1(k,x)&=\int_0^x \sqrt{\frac{1-k^2v^2}{1-v^2}}dv\\
H_1(k,n,x)&=\int_0^x \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}
\end{aligned}$$

called Jacobi's forms for the elliptic integrals of the first, second, and the third kinds respectively. These are complete integrals if $x=1$. The Jacobi's forms conform to the definition of elliptic integral.

In fact, any elliptic integral is a linear combination of elementary functions and the three kinds of elliptic integrals.

By looking at a family of such integration, although we could not obtain the exact value, we can prove

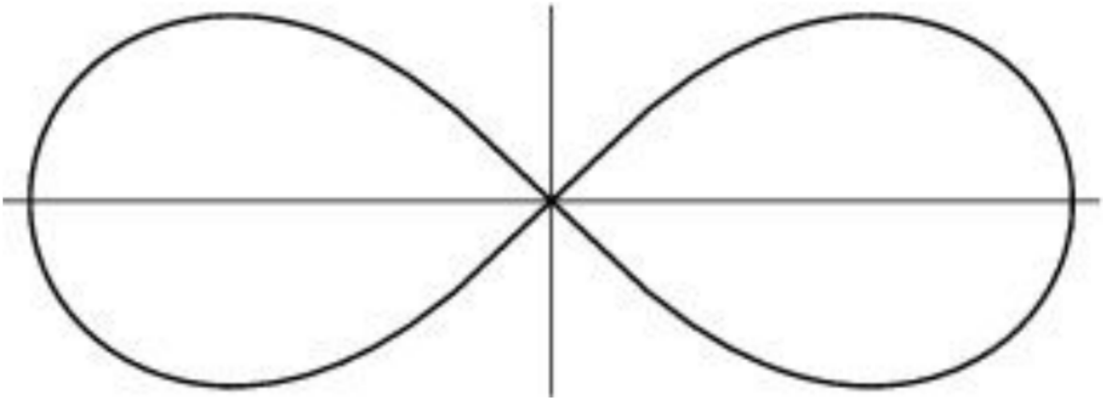

Figure 6.1: Lemniscate

some nice relations between them. The relations are much like the formulas for trigonometric, exponential, or logarithmic functions which also provide many information. We will not discuss these relations but one could refer to http://www.mhtlab.uwaterloo.ca/courses/me755/web_chap3.pdf.

Also one could use the elliptic integral calculator in Mathematica or Matlab to get approximate numerical values.

6.2.3 Arclength of a lemniscate

The lemniscate is the curve: $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, or in polar form $r^2 = a^2\cos 2\theta$. It is the locus of points the product of whose distances from two points (called the foci) is a constant. See figure 6.1

We shall use the arclength formula for polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Then

$$\begin{cases} \frac{dx}{d\theta} = \frac{\partial x}{\partial r}\frac{dr}{d\theta} + \frac{\partial x}{\partial \theta} = \cos\theta r'(\theta) - \sin\theta r(\theta) \\ \frac{dy}{d\theta} = \frac{\partial y}{\partial r}\frac{dr}{d\theta} + \frac{\partial y}{\partial \theta} = \sin\theta r'(\theta) + \cos\theta r(\theta) \end{cases}$$

So

$$\Rightarrow \sqrt{x'(\theta)^2 + y'(\theta)^2}d\theta = \sqrt{r'(\theta)^2 + r(\theta)^2}d\theta$$

$$\begin{aligned} L = 4\int_{\theta=0}^{\pi/4} ds = 4a\int_{\theta=0}^{\pi/4} \frac{1}{\sqrt{\cos 2\theta}}d\theta = \int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos 2\theta}}, \quad \left(\cos 2\theta = \cos^2 u\right) \Rightarrow \\ = \int_0^{\pi/2}\frac{du}{\sqrt{2-\sin^2 u}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{du}{\sqrt{1-\frac{1}{2}\sin^2 u}} = \frac{1}{\sqrt{2}}\cdot K\left(\frac{1}{\sqrt{2}}\right) \end{aligned}$$

Thus, $L = 4a\cdot\frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = a\cdot 2\sqrt{2}(1.85407) = a(5.244102)$.

6.2.4 Arclength of a cubic Bézier curve

Let the curve $(x(t), y(t))$ be defined by polynomials

$$x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$
$$y(t) = b_3t^3 + b_2t^2 + b_1t + b_0$$

The derivatives are:

$$x'(t) = 3a_3t^2 + 2a_2t + a_1$$
$$y'(t) = 3b_3t^2 + 2b_2t + b_1$$

Squaring these equations gives:

$$\begin{aligned}
\left(x'(t)\right)^2 =& \left(3a_3t^2 + 2a_2t + a_1\right)\left(3a_3t^2 + 2a_2t + a_1\right)\\
=&9a_3^2t^4 + 6a_3a_2t^3 + 3a_3a_1t^2 + 6a_3a_2t^3\\
&+ 4a_2^2t^2 + 2a_2a_1t + 3a_3a_1t^2 + 2a_2a_1t + a_1^2\\
=&9a_3^2t^4 + 12a_3a_2t^3 + 6a_3a_1t^2 + 4a_2^2t^2 + 4a_2a_1t + a_1^2\\
\left(y'(t)\right)^2 =&9b_3^2t^4 + 12b_3b_2t^3 + 6b_3b_1t^2 + 4b_2^2t^2 + 4b_2b_1t + b_1^2
\end{aligned}$$

So the calculation of arclength involves t up to the 4th power. A polynomial of fourth order is used to sort this:

$$L(\tau) = \int_0^\tau \sqrt{c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0}dt$$

This is a elliptic integral not of the three kinds. But it can be expressed as a linear combination of elementary functions and the elliptic integral of the three kinds.

6.2.5 Finite-amplitude pendulum.

The equation of motion is:

$$ml\ddot{\theta} = -mg\sin\theta.\text{ Let } p = \dot{\theta} \to p\frac{dp}{d\theta} = -\frac{g}{l}\sin\theta$$
$$\Rightarrow \frac{p^2}{2} = \frac{g}{l}\cos\theta + C.$$

At $t = 0 : \theta = \theta_0, \dot{\theta} = 0 \Rightarrow \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}$. The period, T, is given by

$$\frac{T}{4} = \sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}},$$

or,

$$\begin{aligned}
T &= 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\left(\theta_0/2\right) - \sin^2(\theta/2)}}\\
&= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{du}{\sqrt{1 - k^2\sin^2 u}}, \sin\left(\frac{\theta}{2}\right) = \sin\frac{\theta_0}{2}\cdot\sin u, \quad k = \sin\left(\frac{\theta_0}{2}\right)
\end{aligned}$$

$\therefore T = 4\sqrt{\frac{l}{g}}\cdot K(k)$, an elliptic integral. For the special case of small oscillations, $k = 0$, we get the classical result:

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

6.2.6 Pólya's Random Walk Constants

Let $p(d)$ be the probability that a random walk on a d-D lattice returns to the origin. In 1921, Pólya proved that

$$p(1) = p(2) = 1,$$

but

$$p(d) < 1$$

for $d > 2$. Watson (1939), McCrea and Whipple (1940), Domb (1954), and Glasser and Zucker (1977) showed that

$$p(3) = 1 - \frac{1}{u(3)} = 0.3405373296\ldots$$

where

$$\begin{aligned} u(3) &= \frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dxdydz}{3-\cos x-\cos y-\cos z} \\ &= \frac{12}{\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\{K[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})]\}^2 \end{aligned}$$

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7.1 Supplementary problems in Assignment 9

Problem 7.1 Let D be the parallelogram formed by the lines $x+y=1, x+y=3, y=2x-3, y=2x+2$. Evaluate the line integral

$$\oint_C dx + 3xydy$$

where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

Problem 7.2 Find a potential for the vector field

$$\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j},$$

in the region obtained by deleting the line $(x,0), x \le 0$, from $\mathbb{R}^2$.

Problem 7.3 Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field which is defined in $\mathbb{R}^2$ except at the origin. Suppose that it satisfies the component test $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in positive direction, one has

$$\oint_\gamma Mdx + Ndy = \varepsilon \int_0^{2\pi} [-M(\varepsilon\cos\theta, \varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta, \varepsilon\sin\theta)\cos\theta]d\theta,$$

for all sufficiently small ε. What happens when γ does not enclose the origin?

7.2 Area formula via Green's theorem

Proposition 7.1

Let $D \subset \mathbb{R}^2$ be a domain bounded by the curve γ. Then

$$|D| = \iint_D 1dxdy = \oint_\gamma xdy = \oint_\gamma -ydx = \oint_\gamma \alpha xdy - (1-\alpha)ydx$$

7.3 Discrete vector calculus

In the following we give a discrete version of vector calculus, the recent topic of this course. The discrete version will help us understand the differential forms better. The content of this tutorial is inspired by Lecture 33 of Math 22a Harvard College.

7.3.1 Discrete forms

Let M be any set (You could pick your favorite finite set). Let $M^k = M \times M \times \cdots \times M$ be the direct product of k copies of M.

Definition 7.1

*A **k-form** α on M is a function $\alpha : M^{k+1} \to \mathbb{R}$ satisfying*

$$\alpha(u_1, \ldots, u_i, u_{i+1}, \cdots, u_{k+1}) = -\alpha(u_1, \ldots, u_{i+1}, u_i, \cdots, u_{k+1}).$$

for $u_i \in M, i = 1, \cdots, k+1$ The space of k-form on M is denoted by $\Omega^k(M)$

We have

$$\alpha(u_1, \cdots, u_{k+1}) = 0 \quad \text{if} \quad u_i = u_j$$

and

$$\alpha(u_1, \ldots, u_i, \cdots, u_j, \cdots, u_{k+1}) = -\alpha(u_1, \ldots, u_j, \cdots, u_i, \cdots, u_{k+1}).$$

Example 7.1

1. A zero-form is just a function on M.
2. A one-form α is a function over $M \times M \to \mathbb{R}$ satisfying $\alpha(u, v) = -\alpha(v, u)$. This is the analogy of vector fields.

From a k-form α we could construct a $k+1$-form.

Definition 7.2

*The **exterior derivative** $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the map defined as*

$$\begin{aligned} & d\alpha_k(u_1, \ldots, u_{k+2}) \\ & = \alpha_k(u_1, \cdots, u_{k+1}) - \alpha_k(u_1, \cdots, u_k, u_{k+2}) \\ & + \alpha_k(u_1, \cdots, u_{k-1}, u_{k+1}, u_{k+2}) + \cdots \\ & = \sum_{i=1}^{k+2} (-1)^{k+2-i} \alpha_k(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+2}) \end{aligned}$$

where $\alpha_k \in \Omega^k(M)$ is a k-form and $u_1, u_2, \cdots, u_{k+2} \in M$. By a hat $\hat{u}_i$ on u_i we exclude the i-th argument u_i.

Example 7.2

1. Let α_0 be a zero-form, so it is a function on M. Then

$$\begin{aligned} d\alpha_0(u, v) &= \alpha_0(u) - \alpha_0(v) \\ d : \Omega^0(v) &\to \Omega^1(v) \\ \text{Function} &\xrightarrow{gradient} \text{Vector field.} \end{aligned}$$

2. Let α_1 be a one-form, so it is a function on $M \times M$, then

$$\begin{aligned} d\alpha_1(u, v, w) &= \alpha_1(u, v) - \alpha_1(u, w) + \alpha_2(v, w) \\ d : \Omega^1(v) &\to \Omega^2(v) \\ \text{Vector field} &\xrightarrow{curl} \text{Vector field.} \end{aligned}$$

3. Likewise, the differential of a 2-form is an analogy of taking divergence of a vector field if valued on a tetrahedron (i.e. 4 points).

Proposition 7.2 (Component test)

$d^2 = 0$.

Proof Let α_0 be a zero form. Then $d(d\alpha_0)(u, v, w) = d\alpha_0(u, v) - d\alpha_0(u, w) + d\alpha_0(v, w) = (\alpha_0(u) - \alpha_0(v)) - (\alpha_0(u) - \alpha_0(w)) + (\alpha_0(u) - \alpha_0(w))$.

And you can work out the proof for general cases for any k-form.

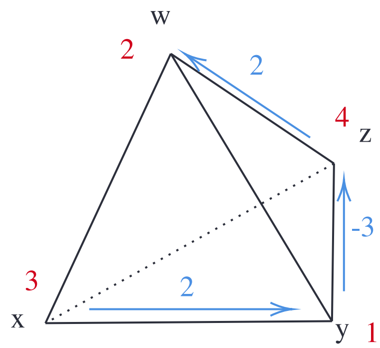

Figure 7.1: $M = \{x, y, z, w\}$

Remark This is the analogy of the following

1. The curl of the gradient of any scalar field φ is always the zero vector field
$$\nabla \times (\nabla \varphi) = \mathbf{0}$$
2. The divergence of the curl of any vector field (in three dimensions) is equal to zero:
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

Pictorially, we could view zero-forms as functions on vertices, one-forms as functions on directed edges, 2-forms as functions on directed triangle faces, 3-forms as functions on directed tetrahedrons, and k-forms as functions on directed k-dimensional simplex. As in figure 7.1, the form on a tetrahedron serves as an analogy of the vector calculus in $\mathbb{R}^3$.

7.3.2 Line integrals and Poincaré's Lemma

Definition 7.3

*A **curve** of M is an element of $\coprod_{k=1}^{\infty} M^k$. If $c \in \coprod_{k=1}^{\infty} M^k$ then we can write it as $c = (u_1, u_2, \cdots, u_m)$ for $u_i \in M, i = 1, 2, \cdots, m$. The curve is **regular** if $\forall i, j, u_i \neq u_j$.* ♣

Definition 7.4

*Let α_1 be a one-form on M, define the **line integration** of α_1 along a curve c as*
$$\int_c \alpha_1 := \sum_{i=1}^{m-1} \alpha(u_i, u_{i+1})$$
♣

Definition 7.5

*A one-form α_1 is called **conservative** if there is a zero-form α_0 such that $\alpha_1 = d\alpha_0$. In this case, α_0 is called the **potential** of α_1.* ♣

Example 7.3 In the figure 7.1 a function f is defined on $M = \{x, y, z, w\}$ as $f(x) = 3, f(y) = 1, f(z) = 4, f(w) = 2$. Then the integration of df along $c = (x, y, z, w)$ is
$$\int_c df = df(x,y) + df(y,z) + df(z,w) = f(x) - f(y) + f(y) - f(z) + f(z) - f(w) = f(x) - f(w) = 1$$

Theorem 7.1 (Fundamental theorem of Line integral)

Let $\alpha_1 = d\alpha_0$ be a conservative one-form and $c = (u_1, u_2, \cdots, u_m)$ be a curve of M, then

$$\int_C \alpha_1 = \alpha_0(u_1) - \alpha_0(u_m)$$

♡

Proof Simple exercise left for the readers.

Therefore, the integration of a conservative 1-form does not depend on the choice of the curves connecting two fixed points.

Proposition 7.3

A one-form α_1 is conserved $\Longleftrightarrow d\alpha_1 = 0$

♠

Proof By proposition 7.2, we just need to prove that $d\alpha_1 = 0$ implies $\alpha_1 = d\alpha_0$ for some zero form α_0. In fact, let u be any point in M, and define $\alpha_0(v) := \alpha_1(v, u)$. Then

$$d\alpha_0(u_1, u_2) = \alpha_0(u_1) - \alpha_0(u_2) = \alpha_1(u_1, u) - \alpha_1(u_2, u)$$

Since $d\alpha_1 = 0$, we have $d\alpha_1(u_1, u_2, u) = \alpha_1(u_1, u_2) - \alpha_1(u_1, u) + \alpha_1(u_2, u)$. Therefore,

$$d\alpha_0(u_1, u_2) = \alpha_1(u_1, u) - \alpha_1(u_2, u) = \alpha_1(u_1, u_2)$$

Proposition 7.3 is not a special phenomenon only for 1-form.

Definition 7.6

*A k-form α_k is **closed** if $d\alpha_k = 0$.*

*A k-form α_k is **exact** if $\alpha_k = d\alpha_{k-1}$ for some $k-1$-form α_{k-1}.*

♣

Theorem 7.2 (Poincaré's Lemma)

A k-form α_k is closed if and only if it is exact.

♡

Proof By proposition 7.2, we just need to show that a closed k-form α_k is exact. Since α_k is closed, it satisfies

$$\begin{aligned} & d\alpha_k\left(u_1, \ldots, u_{k+2}\right) \\ & = \sum_{i=1}^{k+2} (-1)^{k+2-i} \alpha_k\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+2}\right) = 0 \end{aligned}$$

For any $u \in M$, we define $\alpha_{k-1}(u_1, u_2, \cdots, u_k) := \alpha_k(u_1, u_2, \cdots, u_k, u)$. Then

$$\begin{aligned} & d\alpha_{k-1}\left(u_1, u_2, \cdots, u_{k+1}\right) \\ & = \sum_{i=1}^{k+1} (-1)^{k+1-i} \alpha_{k-1}\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+1}\right) \\ & = \sum_{i=1}^{k+1} (-1)^{k+1-i} \alpha_k\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+1}, u\right) \\ & = \alpha_k\left(u_1, \ldots, u_{k+1}\right) \end{aligned}$$

7.3.3 Integration on domains and Stoke's theorem

Mimicking the definition of line integral, we could define the integral of an arbitrary k-form α_k.

Definition 7.7

*A **signed simple domain** D is an ordered tuple of points in M with a sign $+$ or $-$, so we could write it as $D = \pm(u_1, \cdots, u_m), u_i \in M$. We call $m-1$ the **dimension** of D. We define $-D$ to be $\mp(u_1, \cdots, u_m)$. A **domain** V is a set consisting of finite many signed simple domains $\{D_1, D_2, \cdots, D_l\}$, we also write $V = D_1 \vee D_2 \vee \cdots \vee D_l$. Here we allow repetitions, i.e., it may happen that D_i and D_j represent the same signed simple domain. We say V is **pure of dimension** m if D_i is of dimension m for all $i = 1, \cdots, l$.* ♣

Definition 7.8

If $D = (u_1, \cdots, u_m)$ is a signed simple domain, let $F_i = (-1)^{m-i}(u_1, \cdots, \hat{u}_i, \cdots, u_m)$. The signed simple domains F_i are called the facet of D.

*The **boundary** ∂D of a D is defined as*

$$\partial D := F_1 \vee F_2 \vee \cdots \vee F_m.$$

*The **boundary** of a domain V is the disjoint union of the boundary of the signed simple domains in V. By 'disjoint' it means still we allow repetitions, i.e., if a signed simple domain D lies in the boundaries of both D_1 and D_2 of V, then we list D twice in ∂V.* ♣

Definition 7.9

*The **integration** of a k-form α_k over a signed simple domain $D = \pm(u_1, \cdots, u_m)$ is*

$$\int_D \alpha_k = \pm \sum_{1 \leqslant s_1 < s_2 < \cdots < s_{k+1} \leqslant m} \alpha_k\left(u_{s_1}, \ldots, u_{s_{k+1}}\right)$$

*The **integration** of a k-form α_k over a domain $V = D_1 \vee D_2 \vee \cdots \vee D_l$ is*

$$\int_V \alpha_k = \sum_{i=1}^{l} \int_{D_i} \alpha_k$$

♣

Example 7.4 The notation for a curve $c = (u_1, \cdots, u_m)$ of M in definition 7.3 could also be written as $c = (u_1, u_2) \vee (u_2, u_3) \vee \cdots \vee (u_{m-1}, u_m)$, and the line integration for a one-form α_1 over c is the same as the integration of α_1 over the domain $(u_1, u_2) \vee (u_2, u_3) \vee \cdots \vee (u_{m-1}, u_m)$.

Theorem 7.3 (Stokes' theorem)

Let α_k be a k-form and V be a domain which is pure of dimension $k+1$, then

$$\int_V d\alpha_k = \int_{\partial V} \alpha_k$$

♡

Proof We may assume $V = D = (u_1, \cdots, u_{k+2})$ is a signed simple domain of dimension $k+1$.

According to the definition

$$
\begin{aligned}
\int_D d\alpha_k &= \sum_{1 \leqslant s_1 < s_2 < \cdots < s_{k+2} \leqslant k+2} d\alpha_k\left(u_{s_1}, \ldots, u_{s_{k+2}}\right) \\
&= d\alpha_k(u_1, \cdots, u_{k+2}) \\
&= \sum_{i=1}^{k+2} (-1)^{k+2-i} \alpha_k\left(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+2}\right) \\
&= \sum_{i=1}^{k+2} (-1)^{k+2-i} \int_{(u_1, \cdots, u_{i-1}, \hat{u}_i, u_{i+1}, \cdots, u_{k+2})} \alpha_k \\
&= \sum_{i=1}^{k+2} \int_{F_i} \alpha_k \\
&= \int_{F_1 \vee \cdots \vee F_{k+2}} \alpha_k \\
&= \int_{\partial D} \alpha_k
\end{aligned}
$$

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8.1 Shoelace formula

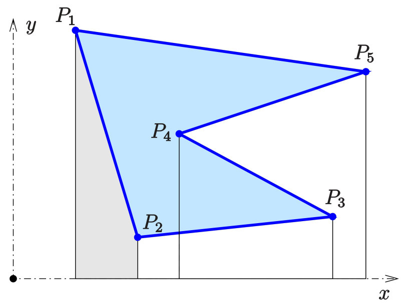

Figure 8.1: A 5-gon

Using Green's theorem, we have a formula to compute the area of a polygon in the plane $\mathbb{R}^2$ if we know the coordinates of the vertices.

A planar simple polygon P is represented by a positively oriented (counter clock wise) sequence of points $P_i = (x_i, y_i)\,, i = 1, \ldots, n$ in the Cartesian coordinate system. For the simplicity of the formulas below it is convenient to set $P_0 = P_n, P_{n+1} = P_1$. Figure 8.1 is an example of a polygon for $n = 5$. Suppose the boundary of P is denoted by ∂P which consists of straight line segments $\overline{P_iP_{i+1}}$.

Theorem 8.1 (Shoelace formula)

$$|P| = \frac{1}{2}\sum_{i=1}^{n}(x_iy_{i+1} - x_{i+1}y_i) = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$$

♡

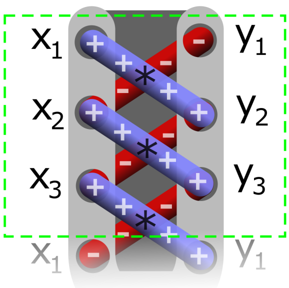

Figure 8.2: Shoelace scheme

Proof The area of P is $|P| = \iint_P 1dxdy$. By Green's theorem,

$$|P| = \iint_P 1dxdy = \frac{1}{2}\oint_{\partial P} xdy - ydx = \frac{1}{2}\sum_{i=1}^{n}\int_{\overline{P_iP_{i+1}}} xdy - ydx$$

Let $c : [0,1] \to \overline{P_iP_{i+1}}$ by $c(t) = (x_i + (x_{i+1} - x_i)t, y_i + (y_{i+1} - y_i)t)$ be a parametrization of the segment $\overline{P_iP_{i+1}}$. Then we have

$$\begin{aligned}
&\int_{\overline{P_iP_{i+1}}} xdy - ydx \\
&= \int_0^1 \left(x_i + \left(x_{i+1} - x_i\right) t\right)\left(y_{i+1} - y_i\right) dt - \int_0^1 \left(y_i + \left(y_{i+1} - y_i\right) t\right)\left(x_{i+1} - x_i\right) dt \\
&= \int_0^1 \left(x_i y_{i+1} - y_i x_{i+1}\right) dt \\
&= \det \begin{pmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{pmatrix}
\end{aligned}$$

Therefore

$$|P| = \frac{1}{2}\sum_{i=1}^{n}\left(x_i y_{i+1} - x_{i+1} y_i\right) = \frac{1}{2}\sum_{i=1}^{n}\begin{vmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{vmatrix}$$

Remark It is hard to generalize the proof to high dimensions to compute volume of polytopes. Given a n-dimension polytope P in $\mathbb{R}^n$. Then Stokes's theorem implies

$$|P| = \int_P 1 dx_1 dx_2 \cdots dx_n = \int_{\partial P} x_1 dx_2 \cdots dx_n$$

Although the computation is reduced to dimension $n-1$, it is still complicated to compute.

On the other hand, there is a direct way via linear algebra. We may assume the facets of P is the union of simplexes by triangulation. Firstly we could compute the volume of a simplex Δ_n spanned by $n+1$ points $P_i = (x_i^1, x_i^2, \cdots, x_i^n), i = 1, \cdots, n+1$ using determinants.

For convenience, let's assume $n = 3$, then

$$\begin{aligned}
\mid \Delta_3 \mid &= \mid \det \begin{pmatrix} x_2^1 - x_1^1 & x_3^1 - x_1^1 & x_4^1 - x_1^1 \\ x_2^2 - x_1^2 & x_3^2 - x_1^2 & x_4^2 - x_1^2 \\ x_2^3 - x_1^3 & x_3^3 - x_1^3 & x_4^3 - x_1^3 \end{pmatrix} \mid \\
&= \mid \det\left(P_2P_3P_4\right) + \det\left(P_3P_1P_4\right) + \det\left(P_1P_2P_4\right) + \det\left(P_2P_1P_3\right) \mid .
\end{aligned}$$

Here $\det\left(P_iP_jP_k\right) = \det \begin{pmatrix} x_i^1 & x_j^1 & x_k^1 \\ x_i^2 & x_j^2 & x_k^2 \\ x_i^3 & x_j^3 & x_k^3 \end{pmatrix}$. The second equality is by multi-linearity of determinant.

By cancellation, the determinant about a common facet of two simplexes does not contribute. So if we assume P is a polytope whose facets are all simplexes, then $|P| = \mid \sum \det\left(P_{i_1}P_{i_2}\cdots P_{i_n}\right) \mid$. The summation is taken over sequences of n vertices that form a facet of P in the order that has a compatible orientation.

8.2 Isoperimetric inequality

Theorem 8.2 (The Isoperimetric Inequality)

Let $c(t) = (x(t), y(t)), t \in [0,1]$ be a simple, closed, positively oriented and regular parameterised curve with $\mathrm{t} \in [a, b]$*. Denote the area enclosed in the above defined curve* $\mathrm{c}(\mathrm{t})$ *with A. Denote the length* *of* $\mathrm{c(t)}$ *by* $l := \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}dt$*, we then have*

$$A \leq \frac{l^2}{4\pi}$$

with equality iff $\mathrm{c(t)}$ *is a circle.*

The theorem comes from the question: Among all closed curves in the plane of fixed perimeter, which curve (if any) maximizes the area of its enclosed region?

Proof

The function $x(t)$ must be bounded. Say $m = \max_{t\in[0,1]} |x(t)|$. We may assume domain bounded by the curve c is convex, and by horizontal shifting we may assume $x'(t) > 0, 0 < t < p; x'(t) > 0, p < t < 1, x(0) = -m, x(p) = m$.

Define a circle by the parametrization $k(t) = (x(t), z(t),\ z(t) = -\sqrt{m^2 - x(t)^2}$ for $0 \leq t < p$ and $z(t) = \sqrt{m^2 - x(t)^2}$ for $p \leq t \leq 1$.

The area $A = \oint_c xdy = \int_0^1 x(t)y'(t)dt$. Let B be the area enclosed by $k(t)$. Then $B = \int_0^1 x(t)z'(t)dt = -\int_0^1 z(t)x'(t)dt = \pi m^2$. Add A to B,

$$\begin{aligned} A + B = A + \pi m^2 &= \int_0^1 (y'x - zx')\,dt \\ &\leq \int_0^1 \sqrt{(y'x - zx')^2}dt \\ &\leq \int_0^1 \sqrt{(x^2 + z^2)\left((x')^2 + (y')^2\right)}dt \\ &= \int_0^1 m\sqrt{x'(t)^2 + y'(t)^2}dt = lm \end{aligned}$$

By mean inquality,

$$\begin{aligned} &\sqrt{A}\sqrt{\pi m^2} \leq \frac{A + \pi m^2}{2} \leq \frac{lm}{2} \\ &\Rightarrow A \leq \frac{l^2}{4\pi} \end{aligned}$$

To get equality, we have $A = \pi m^2 = \frac{1}{2}lm$ and $-xx' = zy'$ for all the inequality above. Squaring we get $x^2(x'^2 + y'^2) = m^2y'^2$. We may assume $x'^2 + y'^2 = l^2$ by choosing a different parametrization. Thus $2\pi x = \pm y'$. Exchanging the role of x and y we got $2\pi y = \pm x'$. Finally

$$x^2 + y^2 = \frac{1}{4\pi^2}(x'^2 + y'^2) = \frac{l^2}{4\pi^2} = m^2$$

So $c(t)$ is a circle of radius m.

8.3 Area of surface of revolution

Problem 8.1 Let S be the surface of revolution obtained by rotating $\mathbf{r}(t) = (f(z), z), f(z) > 0, z \in [a, b]$ around the z-axis. Show that its surface area is given by

$$2\pi \int_a^b f(z)\sqrt{1 + f'^2(z)}dz.$$

Derive this formula using Riemann sum approach.

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9.1 Supplementary problems in Assignment 11

Problem 9.1 Let S be the triangle with vertices at $(1,0,0),(0,2,0),(0,0,7)$ with normal pointing upward. Verify Stokes' theorem for the vector field $\mathbf{F} = x\mathbf{i} + 3z\mathbf{j}$.

Problem 9.2 Let S be the surface given by $(x,y) \mapsto (x,y,f(x,y)),(x,y) \in D$. That is, it is the graph of f over the region D. Show that in this case Stokes' theorem

$$\iint_S \nabla \times \mathbf{F} d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

($\mathbf{F}$ is a smooth vector field on S) can be deduced from Green's theorem for some vector field on D. Hint: Let the boundary of D be $\mathbf{r}(t) = (x(t), y(t))$. Then the boundary of S is $\mathbf{c}(t) = (x(t), y(t), f(x(t), y(t)))$. Convert the integration in S and C to the integration on D and the boundary of D respectively.

9.2 A proof of Brouwer fixed-point theorem [1]

Let $D \subset \mathbb{R}^2$ be the unit disk $D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$. Then the boundary ∂D of D is the unit circle $\mathbb{S}^1 = \{(\cos\theta, \sin\theta) | \theta \in [0, 2\pi]\}$.

Theorem 9.1

Let $f : D \to D$ be a continuous map. There exists $x \in D$ such that $f(x) = x$.

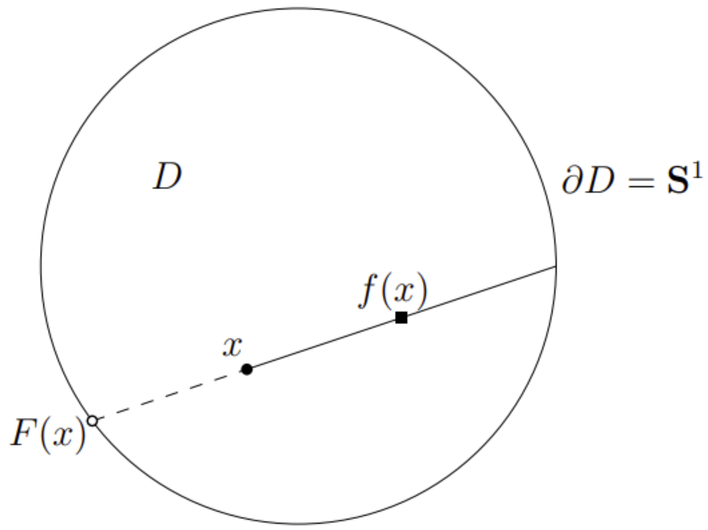

Figure 9.1

Proof We argue by contradiction. If for all $x \in D, f(x) \neq x$, consider a ray starting from $f(x)$ which passes through x after which it has exactly one intersection with ∂D. Let's denote the intersection point by $F(x)$. Therefore, we have a well-defined map $F : D \to \mathbf{S}^1 \subset \mathbb{R}^2, \quad x \mapsto F(x)$. The map F is continuous. We

[1] The reference for this proof is in page 595 of Pin Yu's mathematical analysis, and you could find the book on https://github.com/wuyudi/good-books

could further assume it is continuously differentiable (One will know why we can make this assumption from differential topology).

The map F could be written in terms of coordinates $F(x, y) = (u(x, y), v(x, y)) \in \mathbb{R}^2$. Then $u(x, y) = x, v(x, y) = y$ if (x, y) lies in the circle $\mathbb{S}^1$.

Consider the integration

$$\begin{aligned} I &= \oint_{\partial D} xdy - ydx \\ &= \int_D 2dxdy \\ &= 2\pi. \end{aligned}$$

On the other hand,

$$\begin{aligned} I &= \oint_{\partial D} xdy - ydx \\ &= \int_0^{2\pi} \cos\theta d(\sin\theta) - \sin\theta d(\cos\theta) \\ &= \int_0^{2\pi} u(\cos\theta, \sin\theta)d(v(\cos\theta, \sin\theta)) - v(\cos\theta, \sin\theta)d(u(\cos\theta, \sin\theta)) \\ &= \oint_{\partial D} u(v_x dx + v_y dy) - v(u_x dx + u_y dy) \\ &= \oint_{\partial D} (uv_x - vu_x)\, dx + (uv_y - vu_y)\, dy \end{aligned}$$

We write u_x for $\frac{\partial u}{\partial x}$,etc., for convenience.

Since the image of F lies in $\mathbb{S}^1$, we have

$$u^2 + v^2 = 1 \Rightarrow \begin{cases} u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \\ u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \end{cases}$$

As $(u, v) \neq (0, 0)$, the the determinant for the linear equations is zero,

$$u_x v_y = u_y v_x$$

Therefore,

$$\begin{aligned} I &= \oint_{\partial D} (uv_x - vu_x)\, dx + (uv_y - vu_y)\, dy \\ &= \oint_{\partial D} ((uv_y - vu_y)_x - (uv_x - vu_x)_y) dxdy \\ &= 0, \end{aligned}$$

which is a contradiction.