THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2022-2023 Term 1
Quiz 1 suggested solutions

1. (20 marks) Let $\{a_n\}$ be a sequence of positive real numbers, which is defined by

$$\begin{cases} a_{n+1} = \sqrt{10 + a_n} & \text{for } n \geq 1 \\ a_1 = \sqrt{10} \end{cases}$$

(a) Show that $\{a_n\}$ is bounded and increasing.

(b) Find the limit of $\{a_n\}$.

Solution:

(a) First of all,

$$0 \leq a_1 = \sqrt{10} \leq 10$$

Assume $0 \leq a_n \leq 10$.

$$0 \leq a_{n+1} = \sqrt{10 + a_n} \leq \sqrt{10 + 10} \leq 10.$$

By induction,

$$\forall n \in \mathbb{Z}^+, 0 \leq a_n \leq 10$$

Thus, $\{a_n\}$ is bounded.
Moreover,

$$a_2 - a_1 = \sqrt{10 + \sqrt{10}} - \sqrt{10} \geq 0$$

Assume $a_n - a_{n-1} \geq 0$.

$$\begin{aligned} a_{n+1} - a_n &= \sqrt{10 + a_n} - \sqrt{10 + a_{n-1}} \\ &= \frac{(10 + a_n) - (10 + a_{n-1})}{\sqrt{10 + a_n} + \sqrt{10 + a_{n-1}}} \\ &= \frac{a_n - a_{n-1}}{\sqrt{10 + a_n} + \sqrt{10 + a_{n-1}}} \geq 0. \end{aligned}$$

By induction,

$$\forall n \in \mathbb{Z}^+, a_{n+1} - a_n \geq 0$$

Hence, $\{a_n\}$ is increasing.

(b) Let $A = \lim_{n \to \infty} a_n$.

$$\begin{aligned} A = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{10 + a_n} &= \sqrt{10 + \lim_{n \to \infty} a_n} = \sqrt{10 + A} \\ \implies \quad & A^2 - A - 10 = 0 \\ \implies \quad & A = \frac{1 + \sqrt{41}}{2} \quad \text{or} \quad A = \frac{1 - \sqrt{41}}{2} \ (< 0, \text{rejected}). \end{aligned}$$

Hence, $\lim_{n \to \infty} a_n = \dfrac{1 + \sqrt{41}}{2}$.

2. (15 marks) For each of the given functions $f(x)$, find its natural domain D, that is, the largest subset of $\mathbb{R}$ on which $f(x)$ is defined. Then state, without proof, whether the function $f : D \to \mathbb{R}$ is: **both injective and surjective**, **injective only**, **surjective only**, or **neither**.

   (a) $f(x) = \ln(|x| - 2)$

   (b) $f(x) = \dfrac{3}{\sqrt{4-x}}$

   (c) $f(x) = \sqrt{x^3 - x}$

Solution:

   (a) Note that
   $$|x| - 2 > 0 \iff |x| > 2 \iff x > 2 \text{ or } x < -2$$
   So, we have $D = (-\infty, -2) \cup (2, \infty)$
   With some consideration, it can be seen that f is surjective only.

   (b) Note that
   $$4 - x \geq 0 \text{ and } \sqrt{4-x} \neq 0 \iff 4 \geq x \text{ and } x \neq 4$$
   So, we have $D = (-\infty, 4)$
   With some consideration, it can be seen that f is injective only.

   (c) First of all,
   $$f(x) = \sqrt{x^3 - x} = \sqrt{x\,(x+1)\,(x-1)}$$
   The domain will be given by
   $$x\,(x+1)\,(x-1) \geq 0$$
   That is, $D = [-1, 0] \cup [1, \infty)$.
   With some consideration, it can be seen that f is neither injective nor surjective.

3. (25 marks) Evaluate, without using L'Hospital's rule, the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin 2x}{e^x - e^{-x}}$

   (b) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}}\right)$

   (c) $\displaystyle\lim_{x\to\infty} \left(\frac{x^2 + 5x + 4}{x^2 - 3x + 7}\right)^x$

   (d) $\displaystyle\lim_{n\to\infty} \left(\frac{k}{\sqrt{n^2+1}} + \frac{k}{\sqrt{n^2+2}} + \cdots + \frac{k}{\sqrt{n^2+n}}\right)$, where k is a positive constant.

   (e) $\displaystyle\lim_{x\to 1} \left(\frac{x^{n+1} - (n+1)x + n}{(x-1)^2}\right)$, where n is a positive integer.
   (**Fact:** You may use without proof that $\displaystyle\lim_{x\to a} \left(\frac{x^n - a^n}{x - a}\right) = na^{n-1}$.)

Solution:

(a)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin\ 2x}{e^x-e^{-x}} &= \lim_{x\to 0} e^x\cdot\frac{\sin\ 2x}{e^{2x}-1}\\ &= \lim_{x\to 0} e^x\cdot\frac{\sin\ 2x}{2x\cdot\frac{e^{2x}-1}{2x}}\\ &= \lim_{x\to 0} e^x\cdot\frac{\sin\ 2x}{2x}\cdot\frac{1}{\frac{e^{2x}-1}{2x}}\\ &= 1\end{aligned}$$

(b)

$$\begin{aligned} & \lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)\\ = & \lim_{x\to\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}\right)\cdot\frac{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\\ = & \lim_{x\to\infty}\frac{(x+\sqrt{x})-(x-\sqrt{x})}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\\ = & \lim_{x\to\infty}\frac{2\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x-\sqrt{x}}}\\ = & \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{\sqrt{x}}}+\sqrt{1-\frac{1}{\sqrt{x}}}}\\ = & \ 1\end{aligned}$$

(c) By long division,

$$\frac{x^2+5x+4}{x^2-3x+7} = 1+\frac{8x-3}{x^2-3x+7}$$

Moreover,

$$\lim_{x\to\infty}\frac{x^2-3x+7}{8x-3} = \infty\ (\text{DNE}) \implies \lim_{x\to\infty}\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\left(\frac{x^2-3x+7}{8x-3}\right)} = e$$

$$\lim_{x\to\infty} x\cdot\frac{8x-3}{x^2-3x+7} = \lim_{x\to\infty}\frac{8x^2-3x}{x^2-3x+7} = \lim_{x\to\infty}\frac{8-3/x}{1-3/x+7/x^2} = 8$$

Hence,

$$\begin{aligned}&\lim_{x\to\infty}\left(\frac{x^2+5x+4}{x^2-3x+7}\right)^x\\ =&\lim_{x\to\infty}\left(\left(1+\frac{8x-3}{x^2-3x+7}\right)^{\left(\frac{x^2-3x+7}{8x-3}\right)}\right)^{x\cdot\frac{8x-3}{x^2-3x+7}}\\ =&e^8\end{aligned}$$

(d) Note that

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\leqslant\frac{n}{\sqrt{n^2+1}}$$

because $\sqrt{n^2+1}\leqslant\sqrt{n^2+i}$ for $i\geqslant 1$.
On the other hand, since

$$\sqrt{n^2+n}\geqslant\sqrt{n^2+i}\quad\text{for } 1\leq i\leq n$$

we have

$$\frac{n}{\sqrt{n^2+n}}\leqslant\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}$$

Thus,

$$\frac{n}{\sqrt{n^2+n}}\leqslant\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i}}\leqslant\frac{n}{\sqrt{n^2+1}}$$

Notice that

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}}=1$$

and

$$\lim_{n\to\infty}\frac{n}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n^2}}}=1$$

Since $\lim_{n\to\infty}\frac{1}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{n}{\sqrt{n^2+n}}=1$, by the squeeze theorem, we have

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{\sqrt{n^2+i}}=1$$

and so,

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{k}{\sqrt{n^2+i}}=k.$$

(e) Let

$$\begin{aligned}
f(x)&=\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}\\
&=\frac{x(x^n-1)-n(x-1)}{(x-1)^2}\\
&=\frac{\frac{x(x^n-1)}{x-1}-\frac{n(x-1)}{x-1}}{x-1}\\
&=\frac{x(x^{n-1}+x^{n-2}+\cdots+x+1)-n}{x-1}\\
&=\frac{(x^n+x^{n-1}+\cdots+x)-n}{x-1}\\
&=\frac{(x^n-1)+(x^{n-1}-1)+\cdots+(x^2-1)+(x-1)}{x-1}\\
&=\frac{(x^n-1)}{x-1}+\frac{(x^{n-1}-1)}{x-1}+\cdots+\frac{(x^2-1)}{x-1}+\frac{(x-1)}{x-1}
\end{aligned}$$

Using the provided fact,

$$\lim_{x\to 1} f(x) = n + (n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n+1)}{2}.$$

4. (20 marks) Given that the function

$$f(x) = \begin{cases} x^{\frac{a}{x-1}} & \text{for } x > 1 \\ b & \text{for } x = 1 \\ \cos x & \text{for } x < 1 \end{cases}$$

is continuous over $\mathbb{R}$. Find, without using L'Hospital's rule, the values of a and b, where $a, b \in \mathbb{R}$.

Solution:

First of all, we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \cos x = \cos 1.$$

and

$$\begin{aligned} \lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} x^{\frac{a}{x-1}} \\ &= \lim_{y\to 0^+} (1+y)^{\frac{a}{y}} \quad \text{by letting } y = x-1 \\ &= \lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{az} \quad \text{by letting } z = \frac{1}{y} \\ &= \left(\lim_{z\to\infty} \left(1+\frac{1}{z}\right)^{z}\right)^{a} \\ &= e^a. \end{aligned}$$

Since f is continuous at $x = 1$, we have

$$\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x) \implies \cos 1 = b = e^a$$

Hence, $a = \ln(\cos 1)$ and $b = \cos 1$

5. (20 marks) For all $x \in \mathbb{R}$, define

$$f(x) = \begin{cases} x^3 \cos\left(\frac{4}{x^2}\right) & \text{for } \quad x > 0 \\ 0 & \text{for } \quad x \leq 0 \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) Is $f(x)$ differentiable at $x = 0$? Explain your claim.

(c) Is $f'(x)$ differentiable at $x = 0$? Explain your claim.

Solution:

(a)

$$f'(x) = \begin{cases} 3x^2 \cos\left(\frac{4}{x^2}\right) + 8\sin\left(\frac{4}{x^2}\right) & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(b) First of all, for any $h \neq 0$,

$$0 \leq \left| h^2 \cos\left(\frac{4}{h^2}\right) \right| \leq h^2$$

$$\lim_{h\to 0^+} 0 = \lim_{h\to 0^+} h^2 = 0$$

By squeeze theorem,

$$\lim_{h\to 0^+} \left| h^2 \cos\left(\frac{4}{h^2}\right) \right| = 0$$

and thus,

$$\lim_{h\to 0^+} h^2 \cos\left(\frac{4}{h^2}\right) = 0$$

So,

$$\begin{aligned} Lf'(0) &= \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0, \\ Rf'(0) &= \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0^+} \frac{h^3 \cos(\frac{4}{h^2})}{h} \\ &= \lim_{h\to 0^+} h^2 \cos(\frac{4}{h^2}) \\ &= 0 \end{aligned}$$

Therefore, we have $Lf'(0) = 0 = Rf'(0)$. So, f is differentiable at 0 and $f'(0) = 0$.

(c) By part (a),

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} \left(3x^2 \cos\frac{4}{x^2} + 8\sin\frac{4}{x^2} \right)$$

Let $a_n = \sqrt{\frac{4}{\frac{\pi}{2}+2n\pi}}$, $b_n = \sqrt{\frac{4}{-\frac{\pi}{2}+2n\pi}}$, $n \in \mathbb{Z}^+$. Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$, but

$$\lim_{n\to\infty} 8\sin\left(\frac{4}{a_n^2}\right) = 8 \neq -8 = \lim_{n\to\infty} 8\sin\left(\frac{4}{b_n^2}\right)$$

By sequential criterion,

$$\lim_{x\to 0^+} 8\sin\left(\frac{4}{x^2}\right) \text{ DNE}$$

By part (b), $\lim_{x\to 0^+} 3x^2 \cos\frac{4}{x^2} = 0$. Therefore, $\lim_{x\to 0^+} f'(x)$ DNE.
Hence, $f'(x)$ is not continuous at 0 and thus, not differentiable at 0.