THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
Mathematics Garden
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Deng Yongzhe Suppose $f(x)$ is twice differentiable in $[0,1]$ and $|f''(x)| \leqslant M$,and $f(x)$ get its maxima in $(0,1)$.Try to show:$|f'(0)|+|f'(1)| \leqslant M$.

Lam Chi Yeung Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$|f(x) - f(y)| \leq (x-y)^2$$

for all x, $y \in \mathbb{R}$. Show that f is a constant.

Ng Tsz Ching Integrate $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$.

Chen Yu What is $1 - 1 + 1 - 1 + 1 - 1 + \cdots =$?

Chen Teng rove the following properties of traces.

1. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$;
2. $\text{tr}(kA) = k\text{tr}(A)$;
3. $\text{tr}(A^T) = \text{tr}(A)$;
4. $\text{tr}(AB) = \text{tr}(BA)$.

Liu Beibei Function f satisfies functional equation $f(x+y) = f(x) + f(y) \quad (\forall x, y \in \mathbb{R})$, and f is continuous at $x = 0$, then there is only one solution $f(x) = ax$ satisfying the equation (a is a constant).

Li Hangfan Here are two problems:

1. Integrate $\displaystyle\int \frac{ln}{x^5} dx$.
2. Integrate $\displaystyle\int \frac{2+\sqrt{x}}{3-\sqrt{x}} dx$.

Xu Ang Show that: $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

Choi Ki Kit Answer the following questions:

1. Show that $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$ is not analytic at 0.
2. Evaluate $\iint\limits_{\mathbb{R}^2} e^{-(x^2+y^2)}dxdy$.
3. Let A be a $n\times n$ self-adjoint matrix. Suppose $R(x)=\frac{<Ax,x>}{||x||^2}$, then $\max\limits_{x\neq 0} R(x)$ is the largest eigenvalue of A.

Chan Ho Yuan Answer the following questions:

1. Let

$$A=\begin{bmatrix} 1 & -1 & -2 & 0 \\ -1 & 1 & 3 & 1 \\ -2 & 2 & 7 & 3 \\ 2 & -2 & -6 & -2 \end{bmatrix}$$

   (a) Find $u_1,\ u_2\in\mathbb{R}^4$ such that $\text{span}\{u_1,\ u_2\}=\text{N}(A)$, where $\text{N}(A)$ is the null space of A.
   (b) Find a $u_3\in\mathbb{R}^4$ such that $Au_3=b$, where $b=(3,\ -4,\ -9,\ 8)$.
   (c) Find a $u_4\in\mathbb{R}^4$ such that $Au_4=b$, where $b=(-2,\ 3,\ 7,\ -6)$.
   (d) Show that every $x\in\mathbb{R}^4$ can be written uniquely as a linear combination of $u_1,\ u_2,\ u_3,\ u_4$.

2. (A First Course in Linear Algebra by Robert A. Beezer, P.56, T40)
   Suppose $Ax=b$ is a consistent system of linear equations in which two columns of A are equal. Prove that the system has infinitely many solutions.

Zuo Cheng Show that Let M be a subspace of the Hilbert space H. Let $v\in H$
M and define $\delta := inf\{\|v-w\| : w\in M\}$.(Note that $\delta>0$ since M is closed in H) Then there exists $w_0\in M$ such that:

$$(i)\|v-w_0\|=\delta,\ \text{i.e., there exists a closest point}\ w_0\in Mtov,\ \text{and}$$
$$(ii) v-w_0\in M^{\perp}.$$

Cheng Siu Hong This is to show $\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$ by computing a double integral (using Fubini's Theorem) and elementary calculus techniques such as integration by parts.

Define the improper integral of an improperly integrable function $f(x)$ by

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

(a) Show that $\int_0^\infty \exp(-xy)\sin(x)dy = \frac{\sin(x)}{x}$.

(b) Evaluate $\int_0^a \exp(-xy)\sin(x)dx$.

(c) By using Fubini's theorem and the result of (a) and(b), show that $\int_0^\infty \frac{\sin(x)}{x}dx = \frac{\pi}{2}$.

Wang Chuiji Show that A,B are commutative matrices, and they both can be diagonalized ,then they can be diagonalized simultaneously.

Choi Chi Po For $m = 1, 2, 3, \ldots,$ $n = 1, 2, 3, \ldots,$ let

$$s_{m,n} = \frac{m}{m+n}.$$

Compute

$$\lim_{n\to\infty} \lim_{m\to\infty} s_{m,n}$$

and

$$\lim_{m\to\infty} \lim_{n\to\infty} s_{m,n}.$$

Do they have the same value? Explain your answer.

Wen Jia Find the first five derivatives of the following functions:

1. $f(x) = \dfrac{1}{2-x}$
2. $f(x) = \ln(3+x)$

Yin Guojian Answer the following questions:

1. Using L'Hospital's Rule to evaluate $\lim_{x\to 0} \dfrac{(1-\cos x)\sin 4x}{x^3\cos x}$.
2. Find $\int e^{2x}\sin x dx$.

Luo Tianwen Compute the following limit

$$\lim_{x \to 0^+} x \ln x$$

Liu Xin Prove that: in an n-dimensional real Euclidean space, the operator $\Delta = \Sigma_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ does not change under rotation.

Du Yangge Find the following limit by Riemann integral:

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{k}{n^2} \sin(\frac{k}{n}).$$

Kong Shilei Let a, b, c, d be some real numbers such that the limit

$$\lim_{x \to 0} \frac{\sin^2 2x + a + bx + cx^2 + dx^3}{x^4}$$

exists. Find the values of a, b, c, d and the limit.

Mei Yu Find $\lim\limits_{x \to 0} x^{\sin x}$.

Lee Man Chun Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(Hint: consider $\displaystyle\int_0^1 \int_0^1 \frac{1}{1 - xy} dxdy$).

Min Jie Let A be an $n * n$ matrix and $A * A = A * A^t$. Show that A is symmetric. (Hint: use induction on the dimension of A).

Tao Ran Show that if $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x < y$, then there exists a rational number $p \in \mathbb{Q}$ such that $x < p < y$.

Yuan Zhiri As we can see, for continuous functions, the existence of $\displaystyle\int_0^{\infty} f(x)dx$ and $\lim\limits_{x \to \infty} f$ are somehow related. If $\lim\limits_{x \to \infty} f$ does not equal to zero, then $\int_0^{\infty} f(x)dx$ makes no sense. So, will $\lim\limits_{x \to \infty} f$ equal to zero if $\int_0^{\infty} f(x)dx < \infty$?

Yuan Yuan Considering $s_n = \sum_{k=1}^{n} \frac{1}{k!}$, it is easy to prove $s_n < 2$. So we can define

$$\lim_{n\to\infty} s_n = e. \tag{1}$$

Prove

$$\lim_{n\to\infty} (1 + \frac{1}{n})^n = e. \tag{2}$$

Zhang Pengfei Show that if A, B are two $n \times n$ matrices, then

$$\det\left(\begin{bmatrix} A & B \\ B & A \end{bmatrix}\right) = \det(A+B)\cdot\det(A-B).$$

Xiao Yao Calculate the integral of $\int_{-\infty}^{\infty} e^{-x^2} dx$

Chen Guanheng A certain ecological territory contains S thousands squirrels and R thousands rabbits. Currently,there are 4000 of each species, and the grow rates of the population with respect to time satisfies the following equations:

$$\begin{cases} \dfrac{dR}{dt} = 63R - 3RS, \\ \dfrac{dS}{dt} = 26S - RS. \end{cases}$$

Find the relationship of R and S.

Liu Haixia Function f satisfies functional equation $f(x+y) = f(x) + f(y) \quad (\forall x, y \in \mathbb{R})$, and f is continuous at $x = 0$, then there is only one solution $f(x) = ax$ satisfying the equation (a is a constant).

Liu Keji 1. Determine the <u>domain</u> of the given function

$$f(t) = \frac{t+2}{\sqrt{9-t^2}}$$

2. Find the composite function $f(g(x))$.

$$(1)\quad f(u) = 3u^2 + 2u - 6, \quad g(x) = x + 2,$$
$$(2)\quad f(u) = (u-1)^3 + 2u^2, \quad g(x) = x + 1.$$

3. Find functions $h(x)$ and $g(u)$ such that $f(x) = g(h(x))$.

$$(1)\quad f(x) = (x-1)^2 + 2(x-1) + 3\,,$$
$$(2)\quad f(x) = \frac{1}{x^2+1}\,.$$

4. Write an equation for the line with the given properties.

(1) Through (5,-2) with slope $-\frac{1}{2}$.

(2) Through (1,5) and (3,5).

(3) Through (3,5) and perpendicular to the line $x + y = 4$.

5. Find the indicated limit if it exists.

(1) $\lim\limits_{x\to 5} \dfrac{x^2-3x-10}{x-5}\,.$

(2) $\lim\limits_{x\to -2} \dfrac{x^2-x-6}{x^2+3x+2}\,.$

(3) $\lim\limits_{x\to +\infty} \dfrac{x^2-2x+3}{2x^2+5x+1}\,.$

Ruan Pengfei Prove
$$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{\frac{1}{n}x}=1.$$

Fangqiong JIAN Compute $\int \sec x dx$.

Dai Lipeng Compute $\lim \frac{x^2y^2}{x^3+y^3}$ as $x\to 0, y\to 0$.

Zhao Rui A cyclic curve L is given by polar coordinate $r=1+\cos\theta, 0\le\theta\le\frac{\pi}{2}$ and segment $[0,2]$ on x axis and segment $[0,1]$ on y axis. Find the volume of the solid by rotating L around the x axis.