

# The Use of Spatial Decompositions for Constructing Street Centerlines

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## Abstract

Although national data sets are becoming readily available at low cost, scale usually limits their utility for planning and managing small municipalities. As a result, most communities are faced with the construction of their own municipal Geographic Information Systems (GIS), information systems that are critical in handling land-related activities where high accuracy is essential. Most small municipalities cannot afford to begin by commissioning a large scale *cadastral* map and thus must opt for spatially questionable facsimiles where surveys showing administrative boundaries, property lines and street centerlines are suspect. The accuracy in these data can be enhanced and the results of great value to most city operations. We introduce a new method that applies theoretically based spatial decompositions to automate the generation of street centerlines from spatially corrected block and parcel data. This new centerline data base is a vast improvement over existing data bases for most municipalities.

## I. INTRODUCTION

The relatively young field of Geographic Information Science has had significant impact on the development of techniques to capture, measure, store, manage and analyze geographic information within an urban environment. The amount of data that is captured and catalogued in a municipality is now quite voluminous and increasing each day as government departments go completely digital. The invention of technologies such as Remote Sensing, Global Positioning Systems (GPS) and hand held computers or data loggers, produces more data ready for immediate online access than ever before. Much of the data gathered within the municipality can be geographically related rendering location as a unique data field for relating data that might otherwise go unlinked. However, methods of managing these data are often antiquated and land related data held by one agency or department is frequently inaccessible by another. (Zhou, 1995). For example, some departments use one set of base maps for spatial encoding data while others use a different source, often of different scale and accuracy. Some departments catalogue data by tax map, block and parcel, while others rely on street address. To continue these efforts results in no value added through the integration of data and no synthesis can emerge to better plan and manage the municipality.

A Cadastral map or survey showing administrative boundaries and property lines is usually the most detailed and accurate land information available for a municipality and can provide a large-scale base to which other layers of data can be registered or added.

(Zhou, 1995) Although the Cadastral map may provide the ultimate registration base, it is generated from an accurate land survey and is usually the most tedious and expensive to produce. For most small to medium size municipalities, starting with a digital Cadastre is not a viable option. It is likely the majority of municipal Spatial Decision Support Systems (SDSS) (Malczewski, 1997) that build upon a base of street centerline, block and parcel information, do not require the accuracy of a Cadastral base and can be effectively employing a close facsimile. How then might these municipalities produce a relatively accurate yet cost effective georeferenced data base to facilitate the integration of data and service the majority of their information management needs?

## II. OBJECTIVES

The central objective of this study is to develop a new low cost method which integrates and rectifies non-georeferenced parcel map tiles to an accurate surveyed set of benchmarks and employs theoretically based geometric structures to extract the *form* of a set of points to automatically define and generate street centerlines. This new centerline generator can be considered an alternative to the existing generators which often rely heavily on human operator intervention, leading to long delays during the construction of municipal SDSS and often postponement. Although there have been several attempts to automate the construction of street centerlines (Ladak and Martinez, 1996; Christensen, 1996; East, 1997), prob-

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lems still remain correcting for intersections and cul-de-sacs with most solutions still needing software enhancements. The new method proposed here characterizes the *endoskeleton* (Radke, 1988) of a set of points by exploiting notions of relative proximity and neighborliness. These notions are present in two operations in the method, the construction of the Delaunay Triangulation (DT) and the more rigorous and quantifiable  $\beta$ -*skeleton* which delineates a wide spectrum of possible skeletal structures to construct the street centerline.

Before we propose a method which has foundations in *computational morphology* (Toussaint, 1980), we summarize descriptors of shape that decompose point sets into summary bounding shapes, *shape hulls* (Toussaint, 1980) or *exoskeletons*, and internal line structures referred to as *skeletons* by Toussaint (1980), but better described here as *endoskeletons*. Next the theoretical models that define neighborliness and construct *endoskeletons* as part of the process to delineate street centerlines are developed and explained. Finally, the method developed here is described and applied to a municipal data set in the City of Berkeley, California to demonstrate its sensitivity and robustness. We argue the resultant data base provides an inexpensive but accurate skeleton upon which a robust municipal SDSS can be constructed to serve most of the City's application needs.

### III. SPATIAL DECOMPOSITIONS

The spatial decomposition of data is becoming more common (Li, 1984; Radke, 1988, Okabe et al, 1992) with many tools now packaged in popular software. The locational characteristics of observations are abstracted and encoded as spatial data models with points, lines and polygons (cells) characterizing the spatial extent of data (Goodchild, 1992). Further demarcation within these data models is valuable and can take the form of vertical or horizontal spatial data analysis (Gong, 1994), common practice in the processing of data in geographic information science.

Decomposing a point set into simple line or polygon sets can result in perceptually meaningful shapes or structures that better describe the original point set's morphology. These new structures can provide better descriptors of *form*, or *anthromorphic* decompositions as they are referred to by Pavlidis (1977), between neighboring points. *Essential* or *extreme* points in the pattern can be used to decompose and detect the geometrical properties of the points set under study.

This paper is about understanding the neighborliness

and characteristics of points, lines and polygons to aid in the automatic generation of street centerlines. We undertake a number of spatial processes decomposing polygons to lines and then into points, regenerating simple polygons based on notions of neighborly, decomposing those simple polygons into lines and then essential points, and finally decomposing these essential points into lines which form, for the most part, the street centerlines. Streets can vary in complexity from straight to those with extreme curves. Points demarcating street centers possess the same characteristics which make it difficult for a single shape descriptor to be considered the *best* for all possible applications. We generate a number of solutions that describe the internal set of essential points and form descriptors of the street centerline. The method is robust and can consider a variety of street curves while generating street centerline *best fits*. The method draws from both internal (*Endoskeleton-graphs*) and external (*exoskeleton-hulls*) theoretically based shape descriptors.

#### Exoskeleton-hulls

The simplest *exoskeleton* decompositions of a set of points describe very general geometric constructs. The *minimum bounding box*, *minimum bounding circle* (Freeman and Shapira, 1975; Toussaint and Bhattacharya, 1981) or its generalized *minimum bounding ellipse* (Kirkpatrick and Radke, 1985) all provide a crude first approximation of the global shape of a point set. A more sensitive descriptor of global shape is the convex hull or the minimum convex polygon that contains the entire point set (Toussaint, 1980). A generalization of the convex hull introduced by Edelsbrunner et al (1983) introduces a parameterized notion of a family of  $\alpha$ -*hulls* where shapes, essentially cruder and finer than the convex hull, can be defined.

#### Endoskeleton-graphs

The simplest *endoskeleton* decompositions of a set of points that could be considered a shape descriptor is the *nearest neighbor graph* (NNG) which most often results in an unconnected graph with many spatial subsets. If we connect all the subsets in the NNG with the minimal total edge length, the *minimum spanning tree* (MST) results which can be considered the minimal skeleton of the point set. Increasing the complexity of the link structure and allowing circuit graphs to form, gives the skeleton a more expansive shape by connecting more essential neighbors. One such *endoskeleton*, the *relative neighborhood graph* (RNG) (Lankford, 1969), produces edges linking *relative* neighbors  $V_i$  and  $V_j$  if their *lune*, the region of influence formed by the intersection of two circles of



radius  $d(V_i, V_j)$  and centered at  $V_i$  and  $V_j$ , is empty. A conceptually similar graph, the *Gabriel graph* (GG) (Matula and Sokal, 1980), links *Gabriel* neighbors  $V_i$  and  $V_j$  if their *disc*, the circle of influence with radius  $d(V_i, V_j)/2$  which passes through both  $V_i$  and  $V_j$ , is empty. All of these internal shape descriptors are subsets of the Delaunay Triangulation, as  $NNG \subseteq MST \subseteq RNG \subseteq GG \subseteq DT$  (Figure 1).

The Delaunay Triangulation, the maximal planar description of internal structure in a point set, is a popular decomposition that along with its combinatorial dual, the Voronoi diagram, has had one of the greatest unifying effects of all graphs studied in computational geometry and has many interesting properties and applications (Shamos and Hoey, 1975; Getis and Boots, 1978; Okabe et al, 1992). The DT can be efficiently computed in  $O(n \log n)$  time as it is made up of edges that join all Voronoi neighbors embedded in a plane (Toussaint, 1980). Two points  $V_i$  and  $V_j$ , from a point set  $s$ , are Voronoi neighbors and define an edge of the DT if there exists a point  $x$  in a plane for which  $d(x, V_i) = d(x, V_j) = \min\{d(x, V) \mid V \in s\}$ .

The DT can also be computed, although not as efficiently, using as a generative property the notion of empty neighborhoods similar to those used by the RNG and the GG. Conceptually this method helps

explain our cross street sampling strategy employed in this paper. Like the GG *disc* or circle of influence, each Delaunay triangle is defined by an empty circumcircle (Okabe et al, 1992), an empty circle whose circumference intersects all three points of the Delaunay triangle (Figure 2).

If we generalize the process of using *discs* or empty circles in a plane, these *discs* having neighborly properties similar but not equal to those that construct the DT, we can construct a spectrum of *endoskeletons*, the family of  $\beta$ -*skeletons* (Radke, 1983; Kirkpatrick and Radke, 1985; Radke, 1988). The neighborly properties which generate these *endoskeletons*, produce a spectrum of *skeletons* which include the completely connected graph at one extreme, passing through subgraphs like the RNG and GG, to eventually a graph which equals the point set itself. Since our objective is to construct a *better* descriptor of street centerline which integrates the *essential* points in a street center sample point pattern, it is likely that the acceptable skeleton structure will emanate from a fairly narrow range within the overall spectrum.

*$\beta$ -skeletons* provide a hierarchy of descriptors of internal shape based on measures of neighborliness. We use both a *lune* and *disc* (or circle based method) to search a neighborhood for intervening opportunities in the form of other points from a point set  $V_s$  under-study.

*Circle Based Neighborhoods:* For a given pair of points  $V_i$  and  $V_j$  we can construct a continuous family of neighborhoods  $N(V_i, V_j, \beta)$  based on a pair of circles which pass through both  $V_i$  and  $V_j$ , and are indexed

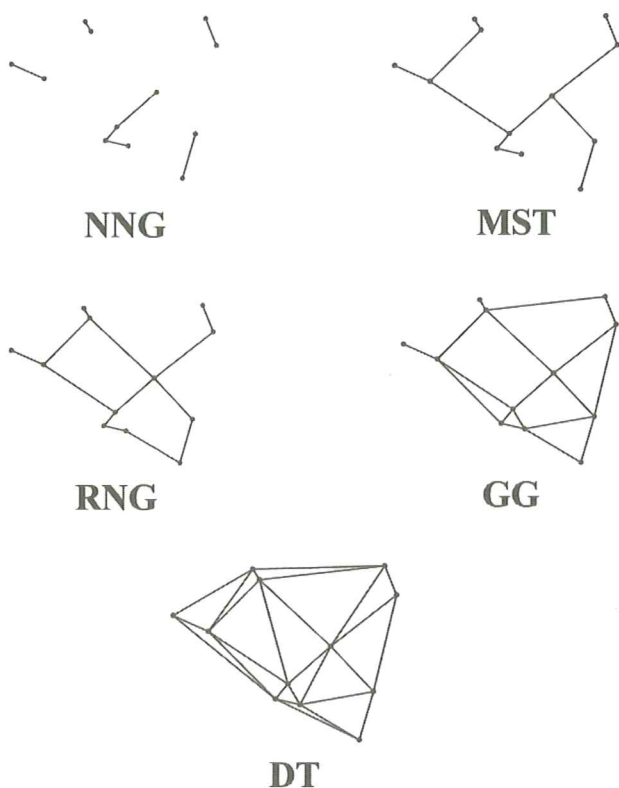


Figure 1.  $NNG \subseteq MST \subseteq RNG \subseteq GG \subseteq DT$

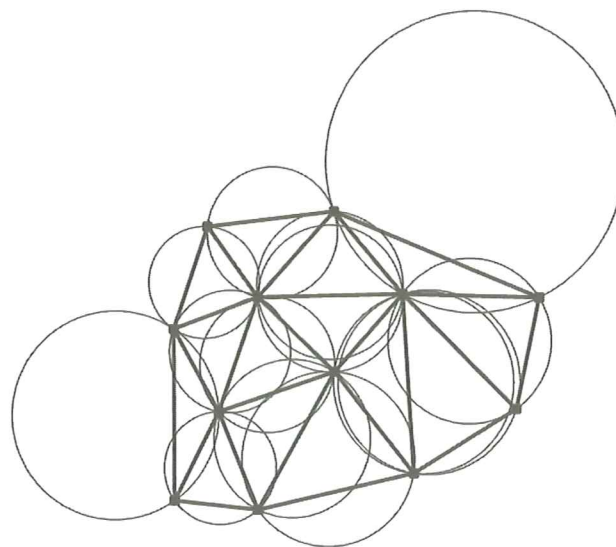


Figure 2. The circumcircle method for generating the DT

by a single real value parameter  $\beta$ , where  $\beta \in [0, \infty]$ . When  $\beta \geq 1$ , the neighborhood searched between  $V_i$  and  $V_j$  is composed of the union of two discs, with radius  $\beta d(V_i, V_j)/2$ , which pass through both  $V_i$  and  $V_j$ . When  $\beta \in [0, 1]$  the neighborhood searched for  $V_i$  and  $V_j$  is the intersection of two discs of radius  $d(V_i, V_j)/(2\beta)$  which pass through  $V_i$  and  $V_j$ . This  $\beta$ -skeleton algorithm generates the *Gabriel graph* when  $\beta = 1$  and the neighborhood searched is  $N(V_i, V_j, 1)$  (Figure 3).



**Figure 4.** City of Berkeley in San Francisco Bay Region.

*Lune Based Neighborhoods:* In the lune based approach we can also construct a continuous family of neighborhoods  $N(V_i, V_j, \beta)$ , indexed by a parameter  $\beta$ , where  $\beta \in [0, \infty]$ . However, when  $\beta \geq 1$ , the neighborhood searched is the intersection of the two circles of radius  $\beta d(V_i, V_j)/2$ , centered at the points  $(1 - \beta/2)V_i + (\beta/2)V_j$  and  $(\beta/2)V_i + (1 - \beta/2)V_j$ , respectively. When  $\beta \in [0, 1]$  the neighborhood searched is the intersection of two discs of radius  $d(V_i, V_j)/(2\beta)$  which pass through  $V_i$  and  $V_j$ . Like the circle based approach, this  $\beta$ -skeleton algorithm generates the *GG* when  $\beta = 1$  but also generates the *RNG* when  $\beta = 2$ .

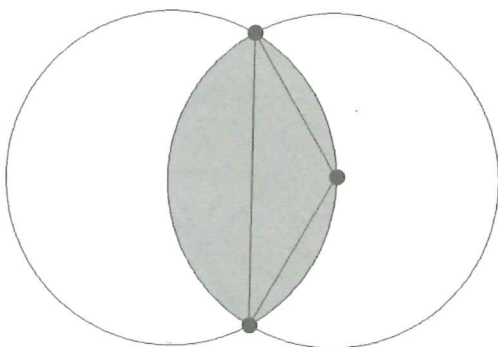
In an effort to develop a functional parcel map which would become the base for a SDSS for the City, we translated a series of Computer Aided Design (CAD) drawings from a local utility company, East Bay Municipal Utility District (EBMUD), into a GIS data structure (Arc/Info). These CAD drawings contained unique water tap numbers within each parcel which are used for billing purposes and eventually lead to a process where we were able to link water tap number and parcel address. From the County Assessment data base we were able to further link parcel address with parcel number (APN) and produce a digital parcel map with both street address and APN, a critical task for the majority of a city's information management needs. When using a variety of ancillary data from different sources, coding errors always exist. After applying some standard quality control measures which included editing the data where necessary, we were able to assess data integrity of the parcel base map to be 97% accurate.

No matter what  $\beta$  method is chosen, as  $\beta \rightarrow \infty$ , the  $\beta$ -skeleton generated, except for degenerate point sets, is devoid of edges and as  $\beta \rightarrow 0$ , the  $\beta$ -skeleton generated becomes the *completely connected graph* (CCG) where edges occur between all pairs of points.

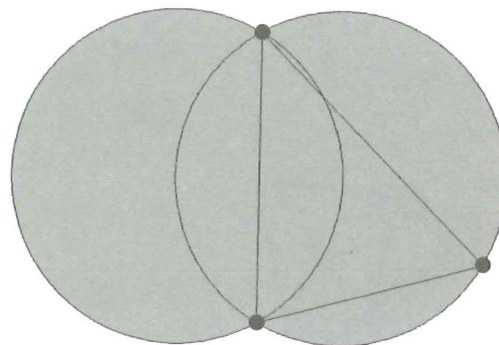
Although the parcels had originally been scanned from

#### IV. DECOMPOSITION METHOD APPLIED

The City of Berkeley is located within the fully urbanized Eastern Shore of the San Francisco Bay in Northern California. With a population of 100,000 the City extends from the Bay east to the Berkeley Hills and is bounded by the City of Oakland to its south and the cities of Albany and Richmond to the north (Figure 4).



$$N(V_i, V_j, \beta) \mid \beta \in [0, 1]$$



$$N(V_i, V_j, \beta) \mid \beta \in [0, \infty]$$

**Figure 3.** Some *disc* or *circle* based  $\beta$ -neighborhoods.

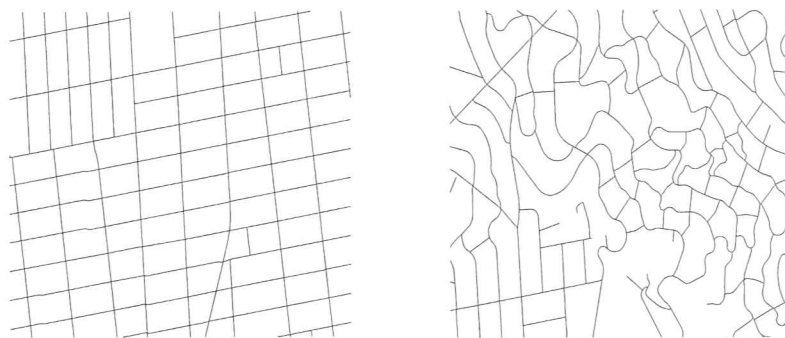


paper cadastral maps, the CAD generated parcel maps were not georeferenced. When the parcel maps were projected into a State Plane Coordinate System using the North American Datum (NAD) 83 and overlaid on 6" panchromatic Aerotopia Digital Ortho photography, we found parcel boundaries transecting both structures and street segments with an average displacement error exceeding 10 feet. To reduce this error we applied rubber sheeting algorithms and adjusted the parcel map to physical, accurately geopositioned monuments, surveyed by the Public Works Engineering Department. Thirty strategically selected monuments, with accuracy within .01 inch, were used as control points and the parcel map georeferenced.

Our process decomposes city blocks or street casings to generate street centerlines. The generation of city blocks in this instance is derived by dissolving the parcel database based on the first seven figures of the APN which are block-unique. This process can return fictitious sliver parcels which will result in a



**Figure 5.** A clean city block database.



**Figure 6.** Southwest Berkeley and Northeast Berkeley street network patterns

contaminated street database with erroneous street segments. The error is eliminated by assigning sliver or multiple polygons within a block an identical attribute value from which we dissolve and produce a clean city block database (Figure 5).

The center of a street is of course the mid point between two opposing blocks. If the blocks are uniform, measuring perpendicular to the block face might suffice, but where the block faces are curved, a more rigorous sampling is needed. In the City of Berkeley the street curvature varies from straight lines in the flat lands to the southwest, to extreme curved lines in the hill areas to the northeast (Figure 6).

To measure the mid point between two opposing blocks we sample along both block faces and compare each sample point to its two closest sample points on the other block face. From this we can easily produce a set of essential or mid points which make up a subset of the street centerline. We conservatively sample every 9 meters along the block face in order to insure we capture the complexity of the curves in the hill area to the northeast (Figure 7).

We construct the Delaunay Triangulation (DT) of the set of points which serves to connect each sample point to its nearest two points on its opposing block face. Of course the DT constructs this connection for the block face across the "street space" as well as the opposing face which constructs the same block (Figure 8).

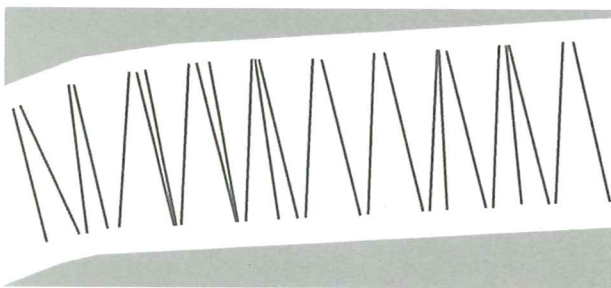
Since our interest is in constructing the mid point across the "street space", we buffer each city block polygon by a very small constant value (in this instance 0.3 meters) and eliminate all line segments that lie within. The resultant data base contains individual line segments (Figure 9) whose mid points lie on and can be used to generate the street centerlines for the entire city (Figure 10).



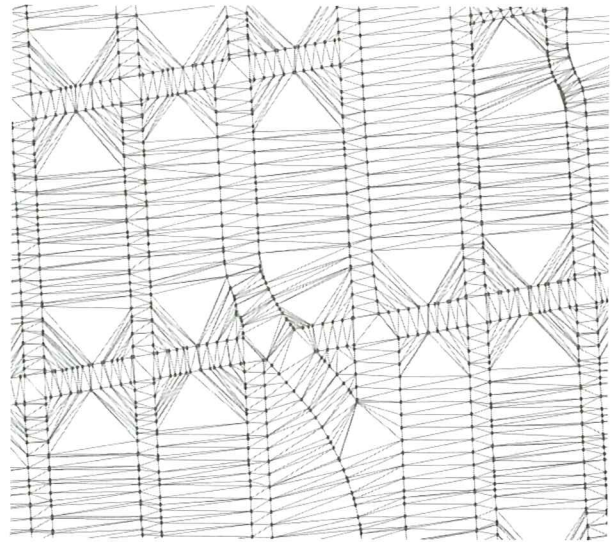
**Figure 7.** Sample every 9 meters along the block face.

It is important to note that in the case of cul-de-sacs the same block face generates points that become Delaunay neighbors which results in essential or mid points that will eventually result in a fork in the centerline termination point. The few cases where this occurred were eliminated before further processing occurred.

What visually appears to be a simple task, to decompose the midpoint database and create street centerlines, is a complex process to automate. If only the *essential* points that sample along the centerline for a given street had to be decomposed, the *minimum spanning tree* (MST) would suffice, however complexity is introduced by the varied and often complicated ways that streets intersect each other. The addition of X, Y and T intersections (Figure 11) call for a decomposition algorithm which can be tuned to integrate the *essential* points in a street center sample point pattern in order to accurately construct street centerlines. We employ both the *Lune* and *disc* based neighborhood methods of the  $\beta$ -skeletons and apply a fairly narrow range within the overall spectrum.



**Figure 9.** The isolated DT line segments crossing a street.



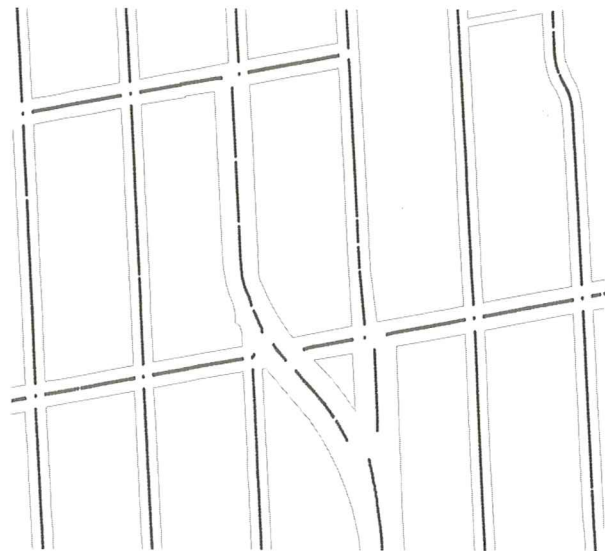
**Figure 8.** DT created from the densified block boundary coverage.

## V. RESULTS

Table 1 contains the results of a number of  $\beta$  values for both the *lune* and *disc* based algorithms.

When  $b=1$ , the *Gabriel Graph* (GG) is generated by both algorithms and a fully connected street network results which easily solves four-way intersections but over connects at T and Y intersections forcing considerable post-processing to create a satisfactory street network.

Although the *disc* based neighborhood appears to be the best and is very effective connecting straight and



**Figure 10.** Resulting street midpoints for a small section of the city.





Figure 11. X, T and Y intersections defined.

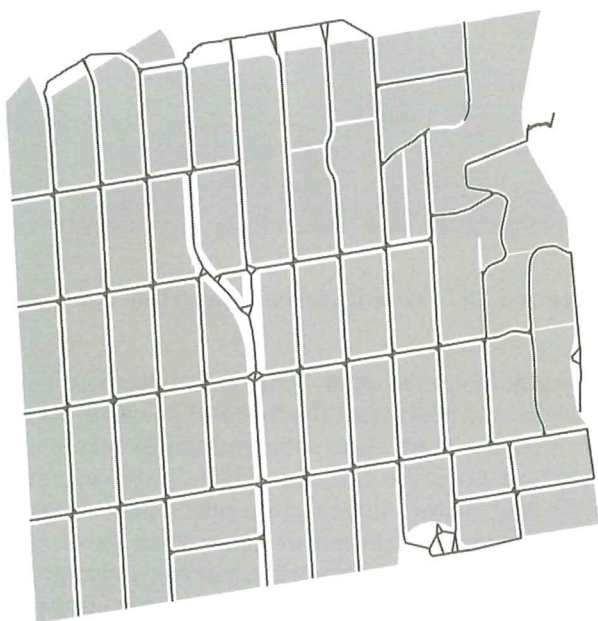


Figure 12.  $\beta = 1$ , the Gabriel Graph (GG) illustrated for a small section of the city.

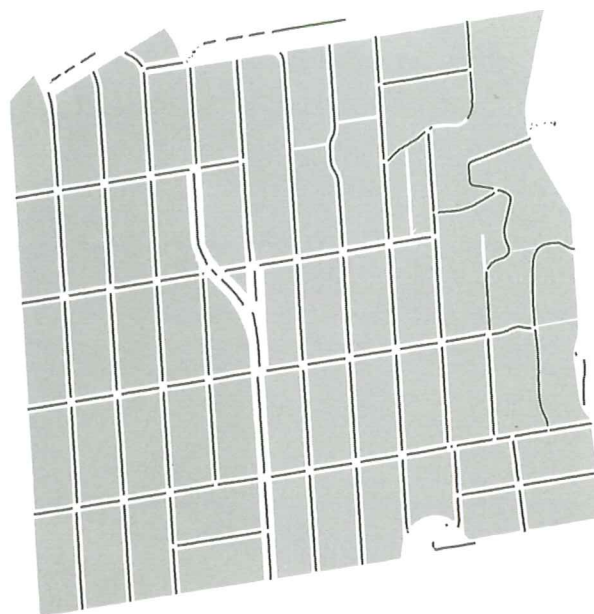


Figure 13.  $\beta > 2$  produces some failed intersections.

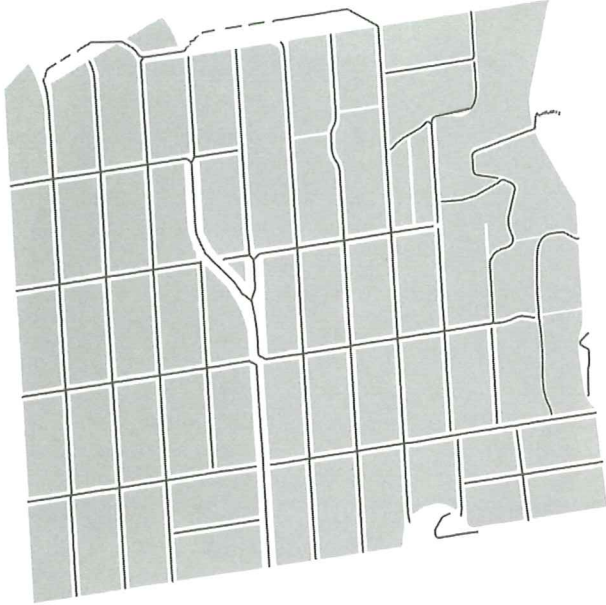
curved line segments, it too produces some failed intersections when  $\beta > 2$  (Figure 13).

After an iterative process we found the circle or *disc* based approach where  $\beta = 1.2$  to produce the most interesting result (Figure 14).

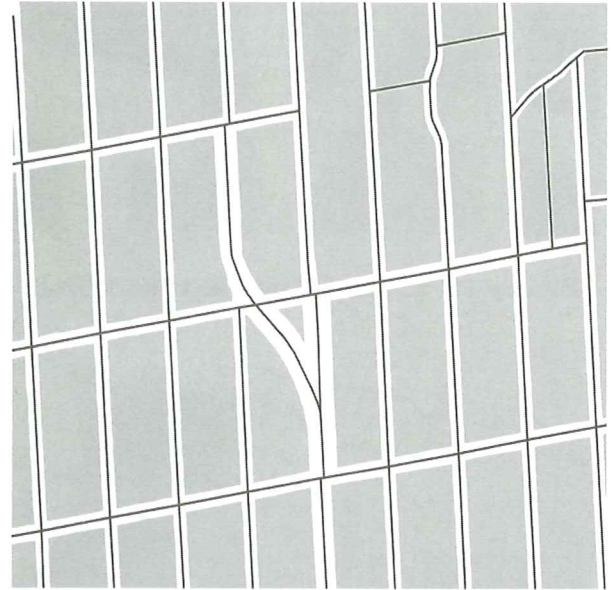
This iteration solves all four-way or X- intersections, Y-intersections and it creates complete arc segments where there are no intersections. The only remaining problem occurs at T-intersections where the algorithm does not properly connect the street center lines.

Table 1. Contains the results of a number of  $\beta$  values for both the *lune* and *disc* based algorithms

$\beta$ -value	Neighbor- hood	X-int.	T-int.	Y-int.	Straight segments	Curved segments	Comment
1.0	Circle	X		X	X	X	T, X & Y overconnect.
1.2	Circle	X		X	X	X	
2.0	Circle				X	X	Straight segments not complete
3.0	Circle					X	Straight segments not complete
4.0	Circle					X	
6.0	Circle					X	
20.0	Circle						
1.0	Lune	X		X	X	X	T, X & Y over-connect.
1.2	Lune	X		X	X	X	T & X over-connect.
2.0	Lune	X		X	X	X	T over-connect.
3.0	Lune			X		X	Straight segments not complete
4.0	Lune					X	Straight segments not complete
6.0	Lune	X				X	Straight segments not complete
20.0	Lune						



**Figure 14.**  $\beta=1.2$  produces the best result.



**Figure 15.** The final street centerline data set.

This anomaly occurs as the original block face sampling to produce the DT and eventually the street centerline point sample set, lacks four block corners at T-intersections and thus can only produce connections that deviate considerably from a right angle.

In post-processing it is possible to automatically extend an arc until it intersects another and corrects the T-intersections. These commands are common editing tools in most GIS where a distance can be set to specify how far the algorithm will search for an intersecting arc. This post-processing step could result in a complete data set but more often a few anomalous errors remain which need attention to ensure that all complicated cases, such as circles and complex intersections are represented correctly. Once the street centerline data set is anatomically correct, we can generalize or omit the redundant vertices in the street centerline data set using weeding algorithms common to most GIS software. Figure 15 illustrates the final street centerline data set processed with a weed tolerance of 2 feet.

## VI. CONCLUSIONS

The central objective of this study, to develop a new method that extracts the *form* of a set of points embedded in a plane and automatically defines and generates street centerlines from parcel-block information, was successfully accomplished. Based on notions of neighborliness, a spectrum of potential centerline solutions are generated which automate the process and provide a better centerline fit, espe-

cially where curves exist. This parameterized notion of neighborliness provides a flexible and powerful method of tuning the centerline construction to better automate and describe the centerline between parcel-blocks and correct intersection problems common to current automated centerline generators. This method is useful in characterizing centerlines where massive parcel-block data sets prevail and automated systems are their only match.

## VII. LIMITATIONS

Although we generalize the process of generating street centerlines, the application is still dependent on the accuracy and completeness of the parcel-block data base. Since we use tax parcel maps as a base data set, right-of-ways are not included. The shape of the block polygons are not always a good representation of how the street is actually aligned. In addition, wide streets with dividers or traffic islands are represented as a single line unless these islands are added in the block coverage.

## ACKNOWLEDGMENTS

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## REFERENCES

- [1] Alnoor, L. and Martinez, R. B. 1996. "Automated derivation of high accuracy road centerlines thiesse polygons technique," in *ESRI User Conference Proceedings*, London, England.
- [2] Aronoff, S. 1989. *Geographic Information Systems: A Management Perspective*. WDL Publications, Ottawa, Canada.
- [3] Burrough, P.A. 1986. *Principles of Geographic Information Systems for Land Resources Assessment*. Clarendon Press, Oxford.
- [4] Christensen, H. A. 1996. "Street centerlines by a fully automated Medial-Axis transformation," *Proceedings of GIS/LIS*, Falls Church, VA., pp. 107-115.
- [5] Donahue, J.G. 1994. "Cadastral mapping for GIS/LIS", *ACSM/ASPRS International Proceedings*.
- [6] East, C. T. 1996. "Automated road centerline generation from double edged road coverages," in *ESRI User Conference Proceedings*, Palm Springs, CA.
- [7] Edelsbrunner, E., Kirkpatrick E.G. and Seidel, R. 1983. "On the shape of a set of points in the plane," *IEEE Trans. Information Theory* 29 (4):551-559.
- [8] Freeman, H. and Shapira, R. 1975. "Determining the minimum-area encasing rectangle for an arbitrary closed curve," *Comm ACM* 18:409-413.
- [9] Getis, A. and Boots, B. 1978. *Models of Spatial Processes: An Approach to the Study of Point, Line and Area Patterns*. Cambridge University Press, New York.
- [10] Goodchild, M.F. 1992. Geographical Information Science, *International Journal of Geographical Information Systems*, 6(1):31-45.
- [11] Gong, P. 1994. Integrated Analysis of Spatial Data From Multiple Sources: An Overview, *Canadian Journal of Remote Sensing*, 20(4):349-359.
- [12] Harary, F. 1969. *Graphy Theory*, Adison-Wesley, Reading.
- [13] Kirkpatrick, K.B. and Radke, J.D. 1985. "A framework for computational morphology," in *Computational Geometry*, Toussaint, G.T., ed., Elsevier Science Publishers B.V. (North Holland), pp. 217-248.
- [14] Lankford, P.M., 1969. "Regionalization: theory and alternative algorithms," *Geographical Analysis* 1 (2): 196-212.
- [15] Li, Yuwei. 1984. Spatial Pattern Recognition by Decomposition, *Mathematical Geology*, 16(3): 217-235.
- [16] Okabe, A., Boots, B. and Sugihara, K. 1992. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. John Wiley & Sons, New York.
- [17] Pavlidis, T. 1977. *Structural Pattern Recognition*, Springer-Verlag, New York.
- [18] Peucker, T.K., Fowler, R.J., Little, J.J. and Mark, D. 1978. "The triangulated irregular network", *Proceedings of the Digital Terrain Models Symposium*, American Society of Photogrammetry, St. Louis, pp. 516-540.
- [19] Radke, J. 1983. "Pattern recognition in circuit networks," Ph.D thesis, Department of Geography, University of British Columbia.
- [20] Radke, J. 1988. A computational geometric approach to the analysis of form. *Computational Morphology*, Toussaint, G.T. (ed) Elsevier Science Publishers B. V. (North Holland), pp 105-136.
- [21] Shamos, M.I. and Hoey, D. 1975. "Closest Point Problems," *16th Annual IEEE Symposium on Foundation of Computer Science*, pp. 151-162.
- [22] Toussaint, G. T. 1978. "The convex hull as a tool in Pattern Recognition," *Proceedings of AFOSR Workshop in Communication Theory and Applications*, Provincetown, MA, pp.43-46.
- [23] Toussaint, G. T. 1980. "Pattern Recognition and geometrical complexity," *Proceedings of the Fifth International Conference on Pattern Recognition*, pp. 1324-1347.
- [24] Toussaint, G.T. and Bhattacharya, B.K. 1981. "On geometric algorithms that use the furthest-point Voronoi diagram," *Technical Report No. socs-81.3*, School of Computer Science, McGill University.