AN INTRODUCTION TO POWER SERIES

A finite sum of the form

$$a_0 + a_1 x + \cdots + a_n x^n$$

(where $a_0, \ldots, a_n$ are constants) is called a polynomial of degree n in x. One may wonder what happens if we allow an infinite number of terms instead. This leads to the study of what is called a *power series*, as follows.

Definition 1. Given a point $c \in \mathbb{R}$, and a sequence of (real or complex) numbers $a_0, a_1, \ldots$, one can form a power series centered at c:

$$a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots,$$

which is also written as

$$\sum_{k=0}^{\infty} a_k (x-c)^k.$$

For example, the following are all power series centered at 0:

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \tag{1}$$

$$1 + x + x^2 + x^3 + \ldots = \sum_{k=0}^{\infty} x^k. \tag{2}$$

We want to think of a power series as a function of x. Thus we are led to study carefully the convergence of such a series. Recall that an infinite series of numbers is said to converge, if the sequence given by the sum of the first N terms converges as N tends to infinity. In particular, given a real number x, the series

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

converges, if and only if

$$\lim_{N \to \infty} \sum_{k=0}^{N} a_k (x-c)^k$$

exists.

A power series centered at c will surely converge at $x = c$ (because one is just summing a bunch of zeroes then), but there is no guarantee that the series will converge for any other values x. Nonetheless, one can prove:

Theorem 1. *Given any power series*

$$\sum_{k=0}^{\infty} a_k (x-c)^k,$$

there exists a number $R \in [0, \infty]$, such that the series converges for all x with $|x-c| < R$, and diverges for all x with $|x-c| > R$.

Such R is clearly uniquely determined by the power series, and is called the radius of convergence of the power series.

For example, the radius of convergence of the power series

$$1 + x + x^2 + x^3 + \ldots$$

is 1, since the series converges for x with $|x-0| < 1$, and diverges for x with $|x-0| > 1$: indeed for any non-negative integer N, the sum of the first $(N+1)$ terms is a finite geometric series, which can be evaluated as

$$1 + x + x^2 + \cdots + x^N = \sum_{k=0}^{N} x^k = \frac{1 - x^{N+1}}{1 - x}.$$

We have that

$$\frac{1 - x^{N+1}}{1 - x} \quad \begin{cases} \text{converges to } \frac{1}{1-x} & \text{if } |x| < 1, \text{ and} \\ \text{diverges if } |x| \geq 1, \end{cases}$$

hence the above conclusion for the radius of convergence of $1 + x + x^2 + x^3 + \ldots$.

We note that we have just established one of the most useful power series in practice:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1 - x} \quad \text{if } |x| < 1.$$

This equality should really be read backwards: in practice, it is often important to know that $1/(1 - x)$ can be expanded into a convergent power series when $|x| < 1$.

Theorem 1 can be proved in full generality by comparing to the geometric series above. We will not enter into the details here.

The radius of convergence of a power series can often be computed using the following theorem:

Theorem 2. *Let R be the radius of convergence of the power series*

$$\sum_{k=0}^{\infty} a_k (x - c)^k.$$

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

$$R = \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|};$$

If $\lim_{n\to\infty} |a_n|^{1/n}$ exists, then

$$R = \frac{1}{\lim_{n\to\infty} |a_n|^{1/n}}.$$

(Here we interpret $1/0$ to be $+\infty$, and $1/+\infty$ to be 0, just so that the statement of the theorem can be more succinct.)

For instance, consider the radius of convergence of the power series (1). There $a_n = \frac{1}{n!}$, and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so the radius of the power series is ∞, i.e. the series converges for all $x \in \mathbb{R}$. This allows one to *define* a function $\exp \colon \mathbb{R} \to \mathbb{R}$, given by

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for all $x \in \mathbb{R}$. We will show that this function has all the properties of our usual exponential function.

Another example is given by the series

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=0}^{\infty} (n+1) x^n. \tag{3}$$

There $a_n = n + 1$, and

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} (n+1)^{1/n} = 1,$$

so the radius of convergence of the series is 1, i.e. the series converges for all x with $|x| < 1$, and diverges for all x with $|x| > 1$.

Note that the above theorem does not say anything about convergence when $|x - c|$ is exactly equal to R. That is something that must be decided case by case.

Also, it can be shown that if $\lim_{n\to\infty} |a_n|^{1/n}$ exists, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, and the two limits are equal. This guarantees that we will get the same radius of convergence for a power series, regardless of the formula we use.

The good thing about staying within the radius of convergence is not just that the series converge, but also that one can differentiate the series term by term:

Theorem 3. *Suppose R is the radius of convergence of the power series*

$$\sum_{k=0}^{\infty} a_k(x-c)^k,$$

and $B_r(c) = \{x \colon |x-c| < R\}$. Then one can define a function f on $B_r(c)$, by

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad \textit{for all } x \textit{ with } |x-c| < R.$$

This function f will be infinitely differentiable on $B_r(c)$; in addition, for all positive integers m, and for all x with $|x-c| < R$, the series

$$\sum_{k=0}^{\infty} \frac{d^m}{dx^m} a_k(x-c)^k \qquad \textit{will converge, and be equal to} \qquad \frac{d^m f}{dx^m}.$$

This is a remarkable theorem, since the infinite sum of differentiable functions may not be differentiable. An example was given by Weierstrass: the series

$$\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k \cos(9^k \pi x)$$

converges for all real numbers x, thereby defining a function $f(x)$ on $\mathbb{R}$, but while f is continuous at every point of $\mathbb{R}$, it is known (but the proof is beyond the scope of this course) that f is not differentiable at any point on $\mathbb{R}$. (This is an example of a function that is everywhere continuous, but nowhere differentiable!)

To illustrate the power of this theorem, one can use this to differentiate $\exp(x)$, and prove that

$$\frac{d}{dx}\exp(x) = \exp(x) \quad \text{for all } x \in \mathbb{R}.$$

Also, recall we knew

$$1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1.$$

The above theorem allows us to differentiate both sides, and obtain, successively,

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \frac{1}{(1-x)^2} \quad \text{for } |x| < 1$$

$$1 + 3x + 6x^2 + 10x^3 + 15x^4 + \cdots = \frac{1}{(1-x)^3} \quad \text{for } |x| < 1$$

etc. (These are remarkable identities! For instance, now we can compute the value of

$$1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \frac{5}{3^4} + \dots.$$

according to the formula for $1/(1-x)^2$, this is just $1/(1-(1/3))^2 = 9/4$, and this is not obvious at all without using differential calculus!)

Another consequence of the above theorem is:

Corollary 4. *Let $f(x)$ be as in Theorem 3. Then the Taylor polynomial of f up to order n at c (which we will denote by $T_{n,c}f(x)$) is just the sum of the first $(n+1)$ terms of the series defining f, i.e.*

$$T_{n,c}f(x) = \sum_{k=0}^{n} a_k(x-c)^k.$$

The proof of the corollary is very easy: one just notes that by the theorem, we have

$$f^{(k)}(c) = k!a_k,$$

so

$$T_{n,c}f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = \sum_{k=0}^{n} a_k(x-c)^k.$$

This gives us a very powerful tool in computing Taylor polynomials.

For example, here is how we would compute the Taylor polynomial of $\exp(x^2)$. We knew

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

so

$$\exp(x^2) = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$$

It follows that the Taylor polynomial of $\exp(x^2)$ up to order 6 at 0 is

$$1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!}.$$

(It would be quite painful to prove this, by directly computing the derivative of $\exp(x^2)$ up to order 6!)