Hints for Assmt 4

Exercise

1. (An illustration of the 'Method of Finding Limit Using Taylor's Theorem)
   Question. Find the limit
   $$\lim_{x\to\infty} \frac{\tan x - \sin x}{\sin^2 x}, \tag{1}$$
   Solution: Many ways to do it. We just mention the Taylor's Theorem Approach. If we use Taylor's Theorem, we use the approximation
   $$\tan x - \sin x \sim \frac{1}{2}x^3, \tag{2}$$
   when $x \to 0$.
   (which can be made rigorous but I don't do it now!)

   Roughly speaking, in the above approximation (i.e. formula (1)), we are expanding the function about the center $c = 0$ up to the degree 3 term!

   Remark Note that if we use Taylor's Theorem on $\tan x - \sin x$ (i.e. formula (2)) about the center $c = 0$, we don't have the degree 0 (i.e. 'constant') term, the degree 1 term and the degree 2 term (i.e. the x and the x^2 terms). They all have coefficients equal to zero! Thus, the first non-zero term of the Taylor's Polynomial (centered at 0) is
   $$\frac{1}{2}x^3.$$

   Similarly,
   $$\sin x \sim x \text{ (when } x \to 0),$$
   (Again, this can be made rigorous, which I will discuss at another time)

Then, we can compute the limit by

$$\lim_{x \to 0} \frac{\tan x - \sin x}{\sin^3 x} = \lim_{x \to 0} \frac{\frac{1}{2}x^3}{x^3} = \frac{1}{2}.$$

2. A good idea to find Taylor polynomial without differentiating a lot is to make use of

   (i) $\frac{1}{1-x} = 1 + x + x^2 + \cdots$

   or

   (ii) 'long-division'. For example

$$\frac{x-1}{x^2+1} = -1 + x + x^2 - x^3 - x^4 + \cdots$$

   This can be seen using 'long-division' as follows:

$$\begin{array}{rl|lllll}
 & & -1+ & x & +x^2 & -x^3 & -x^4 \ \cdots \\
\hline
1 + 0x + x^2 &) & -1+ & 1x & +0x^2 & +0x^3 + \cdots & \\
 & & -1+ & 0x & -x^2 & & \\
\hline
 & & & x & +x^2 & & \\
 & & & x & & x^3 & \\
\hline
 & & & & x^2 & -x^3 & \\
 & & & & x^2 & & +x^4 \\
\hline
 & & & & & -x^3 & -x^4 \\
 & & & & & \cdots &
\end{array}$$

> <u>Remark</u> Of course, the approach (i) is slightly more rigorous. Approach (ii), though it is in a way 'less rigorous', let one 'see' the answer more quickly. It can also be justified if one works carefully.

3. Question 3 is very similar to our proof of the 'second derivative test'. It uses the following 'alternative way of describing the Lagrange's Mean Value Theorem'.

(LMVT)

$$f(x) - f(x_0) = f'(\xi) \cdot (x - x_0)$$

for some ξ between x and the <u>center</u> x_0.

<u>Remark</u> The center dot, i.e. '$\cdot$' means 'multiply' or 'times'. For example $a \cdot b = a \times b$.

If we denote by h the expression $x - x_0$, (i.e. $h = x - x_0$), then LMVT can be written in the form (assuming $x_0 < x$):

$$f(\underbrace{x_0 + h}_{x}) - f(x_0) = f'(\underbrace{x_0 + \theta \cdot h}_{\xi}) \cdot \underbrace{h}_{x - x_0},$$

because now $x = x_0 + h$, and any point ξ between x_0 and x is given by '$x_0 + \theta \cdot h$',
(where θ is a number between 0 and 1).

Now putting the term $f(x_0)$ to the right-hand side of the equal sign, we obtain

$$f(x) = f(x_0) + f'(x_0 + \theta \cdot h) \cdot h.$$