Notes 1 (Lectures in first three weeks)

Introduction

This course 'University Mathematics' is mainly about Calculus. Calculus is actually the short form of 'Differential and Integral Calculus'. As the title suggests, we will discuss two main topics in this course, i.e.

- Differentiation;
- Integration.

Objects of interest

Just as in any mathematics course, there are 'mathematical' objects we are particularly interested in looking at. In primary school math, we have encountered Arithmetic, and there the primitive/atomic/elementary objects (let's just call them 'elementary objects' in the following discussions) we looked at were numbers. So we started with natural numbers, i.e. numbers starting from 0, and then 1, 2, 3, $\cdots$ and from them we built other kinds of numbers using some kinds of 'operations' such as addition, subtraction, multiplication and division (provided you are allowed to do so!)

In secondary school, we learned Geometry, in which the elementary objects were points. From points we built lines, triangles etc.

In Calculus, the elementary objects are something called 'functions'. In school math, we have seen examples of a lot of functions, but most of us haven't seen the 'abstract' definition of these objects.

So following this tradition, let's look at some examples first, and after that look at the definition.

Examples of Functions

Expressions like $2x^2+5$, x^3+x, $\sin(x)$, $\cos(x)$, $(x-1)/(x^2+2)$ are related to functions. (I write 'related' here because we haven't yet made precise what functions are, so there are still room for arguing whether what we have written above are functions or not.)

(Abstract) Definition of Function

A function (with the name f) is a <u>rule</u> which <u>assigns</u> a <u>unique</u> output (written $f(x)$) to each input x.

If we go back to the examples above, the function

$$\cos$$

is a rule which assigns the output $\cos(x)$ to each input x into it.

Another example is the 'absolute value' function

abs (meaning 'absolute value')

which assigns the value x to each input x satisfying $x \geq 0$ and the value $-x$ to each input x satisfying $x < 0$.

Many mathematics textbooks like to use the following diagram to represent such ‘rules’:

$$x \to \boxed{f} \to f(x)$$

If we this kind of representation, the ‘cosine’ would take the form

$$x \to \boxed{\cos} \to \cos(x)$$

while the ‘absolute value’ function would take the form

$$x \to \boxed{\mathbf{abs}} \to \mathbf{abs}(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Remark.
After reading our definition of functions, you should now realize that the ‘expressions’ such as $\mathbf{abs}(x)$ and $\cos(x)$ is ‘values’ of the function

$$\mathbf{abs}$$

and the function

$$\cos$$

for the input x.

First Examples of Functions

Let’s fix some notations before proceeding.

Since our notation of function seems a little naive, let’s make it more sophisticated. To this end, we write

$$x \overset{f}{\mapsto} f(x)$$

to mean the function f sending x to the value $f(x)$. If we are interested in the domain rather that the ‘input’ x as well as the ‘output’ $f(x)$, then we write

$$D \overset{f}{\to} \mathbb{R}.$$

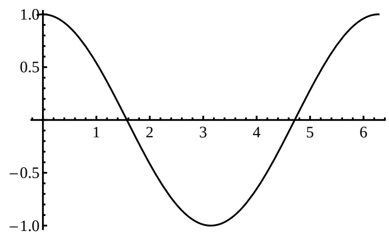
Example If we want to define the cosine function on the domain $[0, 2\pi]$, then we have

$$x \overset{\cos}{\mapsto} \cos(x)$$

and (if we want to talk about the domain only), we have

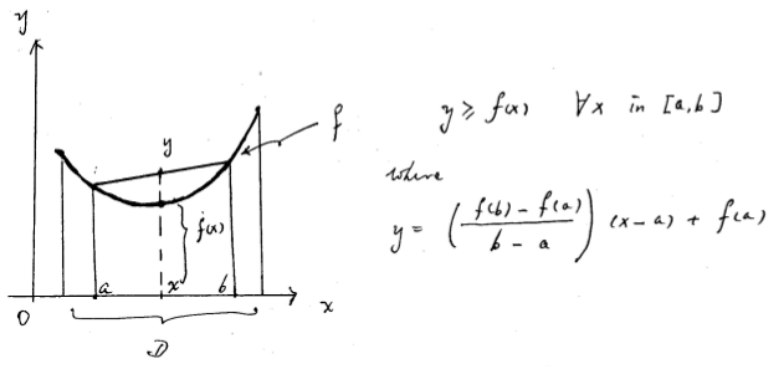
$$[0, 2\pi] \overset{\cos}{\to} \mathbb{R}$$

and the ‘graph’ ([1]) is:



[1] Intuitively, the ‘graph’ of a function is the ‘curve’ defined by this function in the plane.

Another example: convex function
Let f be a function defined on the domain $\mathbb{R}$ (we choose this domain for the reason of simplicity only), then it is called 'convex' if for any a, b satisfying $a < b$ in the domain (i.e. $\mathbb{R}$), the 'line segment' joining $(a, f(a))$ and $(b, f(b))$ lies above the 'graph' [2] of f over $[a, b]$. The following diagram explains what is meant.



Examples of Convex Functions Any function given by $f(x) = x^{2n}$, where n is a natural number gives a convex function over any interval $[a, b]$.

Quantifiers and their Notations

In our definitions so far, we have used the phrase 'for each' a lot. The phrase 'for each' (and similarly, 'for every', 'for all', 'for any') are all denoted by the notation

$$\forall.$$

For examples, if we want to say 'for all real nos. a, b satisfying $a < b$', we can write it as

'$\forall$ real nos. a, b satisfying $a < b$', or

'$\forall a, b$ in $\mathbb{R}$ satisfying $a < b$', or

'$\forall a, b \in \mathbb{R}$ satisfying $a < b$', or even

(where the symbol '$\in$' means 'belongs to' or 'in').

'$\forall a, b \in \mathbb{R}$: $a < b$'

to mean the same thing.

Another related notation people often use is

$$\exists\ ,$$

which means 'for some' or 'there exists'. For example, if we want to say

[2] Here " " (i.e. 'quote and quote') means we are thinking about some kind of 'graph' of f and not the graph of the function f defined over the entire domain D

'Consider the sine function defined over the domain $\mathbb{R}$, for any x, there is a real number bigger than or equal to $|\sin(x)|$',

we can write it as

Consider the sine function, then $\forall x \in \mathbb{R}$, $\exists M \in \mathbb{R}$ s.t. $M \geq |\sin(x)|$. ('s.t.' means 'such that')

Quantifiers The above-mentioned notations are notations for the 'existential' ($\exists$) and the 'universal' ($\forall$) quantifiers.

Using quantifiers, we can rewrite our definition of convex function as:

Definition of Convex Function Let $[\alpha, \beta] \overset{f}{\mapsto} \mathbb{R}$ be a function. Then f is convex if

$\forall a, b$ in $[\alpha, \beta]$ satisfying $a < b$, we have $f(x) \leq \left(\frac{f(b)-f(b)}{b-a}\right)(x-a) + f(a)$, $\forall x$ in $[a, b]$.

Other Examples of Functions Some useful functions we often use are:

- (constant function) These are functions having the same (constant) value for any input x.

  Example.
  $\mathbb{R} \xrightarrow{f} \mathbb{R}$ given by $f(x) = c$ for any given real no. c.

  Graphs of Const. Fns.
  The 'graphs' of such functions are straight lines parallel to the $x-$axis

- (polynomial function) They are finite sum of the form

  $$a_0 + a_1 x^1 + \cdots + a_n x^n.$$

  where n is a non-negative natural integer. The largest power n for which a_n is non-zero is called 'the degree' of the polynomial.

  Example.
  The function $\mathbb{R} \xrightarrow{f} \mathbb{R}$ given by

  $$f(x) = 1 + 2x + x^3 - 4x^5$$

  is a polynomial of degree 5.

- (rational function) One can show that polynomials behave like integers. Now a rational number is simply the number of the form

  $$\frac{\text{integer}}{\text{non-zero integer}}.$$

  Motivated by this, we defined a rational function to be the quotient of two polynomials:

  Examples $\dfrac{x+1}{x^2-3x+2}$, $\dfrac{x^2+x-1}{x^4+1}$. (For the first of these two examples, one has to have domains avoiding the 'zeros' of the denominator, i.e. $x^2 - 3x + 2$)

- (trigo. function) These are the sine, cosine, tangent, cotangent, secant, cosecant functions. They can be defined by starting with the right-angled triangles. Another definition is via 'infinite sum' done as follows (remember always that here x is measured in 'radian'!):

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \end{aligned}$$

- (the $\exp(x)$ function) The exponential function is defined by the 'infinite sum':

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- (exponential functions) Let a be any real number, then we can define exponential functions by

$$f(x) = a^x$$

  As we can see from their graphs, each of these functions is an onto and 'strictly increasing' (if $a > 1$) or 'strictly increasing' (if $a < 1$) function.

By 'strictly increasing', we mean the following.

Definition Let $\mathbb{R} \xrightarrow{f} \mathbb{R}$. (we chose $\mathbb{R}$ as domain, but one can choose any intervals as domain too). The we say f is strictly increasing if for any real nos. a, b satisfying $a < b$, we have $f(a) < f(b)$.

A very important example of the exponential function is the exp function, which is defined by choosing $a = e$, where e is the number

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

we have actually

$$\mathbb{R} \xrightarrow{\exp} (0, \infty)$$

i.e. the set $(0, \infty)$ contain all values of **exp** ! (in other words: 'each point y in $(0, \infty)$ originates from an x in $\mathbb{R}$'). Because of this, one can go back and define from

$$x \xmapsto{\exp} f(x) = y$$

an 'inverse' function defined by

$$x \xleftarrow{\ln} y$$

which does the following: it sends using the rule

ln

the number y (which is equal to e^x (or $\exp(x)$) back to the number x. More precisely, it is:

$$\ln(\quad \exp(x) \quad) = x$$

Hence it actually is 'undoing' what the rule **exp** has done to x (this rule **exp** sends x to $\exp(x)$ $(= y)$) by sending the output $\exp(x)$ back to x, hence the name 'inverse function').

Onto functions, Strictly increasing functions, Inverse function

One very important reason why we talked about the set $(0, \infty)$ for the **exp** function is because this set (which is a subset of $\mathbb{R}$) is the smallest set for which all the values of **exp** fall in, i.e.

$$\forall y \in (0, \infty) \quad \exists x \in \mathbb{R} \text{ s.t. } \exp(x) = y$$

(This statement, though obvious, is not easy to prove!) In such case, we say that the function **exp** is 'onto' (or 'surjective' which is also a name for onto functions used in textbooks).

The onto-ness of the function **exp** together with the fact that it is strictly increasing gives rise to the inverse function **ln**, which is the function we just mentioned.

Let's write the above down as a proposition.

Proposition Let $(a, b) \overset{f}{\mapsto} (c, d)$ (here we chose (a, b) as the domain, but we could have chosen other kinds of intervals as domain of f, such as the interval $[a, b]$ or the interval $[a, b)$ etc.). If f satsifies

(i) f is onto, and

(ii) f is strictly increasing,
(same if we replace 'strictly increasing' by 'strictly decreasing')

then f has an inverse function with domain (c, d).

Examples The above-mentioned function, i.e. **exp** function is such an example.

(Note: Here and in the following, we will write

$$(a, b) \overset{f}{\to} (c, d)$$

as well as

$$f : (a, b) \to (c, d)$$

interchangeably to mean the same thing).

Other Examples

E.g.1 $f : (0, 1) \to (0, 1)$ given by $f(x) = x^2$

E.g.2 $g : (-1, 0) \to (0, 1)$ given by $f(x) = x^2$.

Both of example 1 and example 2 are onto functions (though their domains differ!). The first one (as seen from the graph) is strictly increasing, while the second one is strictly decreasing.

Positive Derivative & Onto

The above example of **exp** functions shows that

(i) ontoness, and

(ii) strictly increasing/decreasing property

are two important conditions to ensure us that we can find inverse function of a given function.

Question

'how can we check (i) ontoness; (ii) strictly increasing/decreasing ?'

Answer

(i) To check onto-ness, what one needs to do is given any y (in (c, d) say) one find an x in the domain ((a, b) say) which solves the equation

$$f(x) = y$$

(this step usually needs a lot of work!)

(ii) To check strictly increasing/decreasing property, one has the following result:

Theorem Let $f : (a, b) \to (c, d)$ (not necessarily 'onto'. Also, we can consider other kinds of intervals instead of (a, b) and (c, d)) satisfy

$$\left.\frac{df(x)}{dx}\right|_{x=c} > 0, \quad \forall x \in (a, b),$$

then f is 'strictly increasing' in (a, b).

Remark If we change '$>$ 0' to '$<$ 0' in the above theorem, then 'strictly increasing' should be replaced by 'strictly decreasing'.

Application If in the above theorem, in addition the function f is actually an onto function, then it has an inverse function $g : (c, d) \to (a, b)$.

Examples

(i) Let $f : (0, 2) \to (0, 4)$ be given by $f(x) = x^2$, then (i) f is 'onto', (ii) $f'(x) > 0,\ \forall x \in (0, 2)$, implying that there exists an inverse function $g : (0, 4) \to (0, 2)$.

(ii) Let $f : (-2, 0) \to (0, 4)$ be given by $f(x) = x^2$, then (i) f is 'onto', (ii) $f'(x) < 0$, $\forall x \in (-2, 0)$, implying that there exists an inverse function $g : (0, 4) \to (-2, 0)$.

Summary

In the above, we have seen a theorem which gives us a condition needed when we want to find the inverse of a function. When describing this theorem we need the concept of derivative. To prove the theorem, we will need to learn something known as 'Mean Value Theorem'.

But before we can prove the theorem, we first need to study the definition of derivative using the concept of limit.