MATH1010 Further Exercise on Integration

1. Evaluate the definite integrals below:

   (a) $\int_0^4 |x(2-x)|dx$ (b) $\int_0^2 |x^2-3x+2|dx$ (c) $\int_{-1}^2 |x^3-x^2-2x|dx$

2. Evaluate the definite/indefinite integrals below:

   (a) $\int x(x^2+2)^{99}dx$ (b) $\int_3^4 \frac{x}{\sqrt{25-x^2}}dx$ (c) $\int \frac{x}{\sqrt{3x^2+1}}dx$ (d) $\int_0^2 \frac{x^2}{\sqrt{9-x^3}}dx$

   (e) $\int x(x+2)^{99}dx$ (f) $\int_1^5 \frac{xdx}{\sqrt{4x+5}}$ (g) $\int x\sqrt{x-1}dx$ (h) $\int (x+2)\sqrt{x-1}dx$

   (i) $\int \frac{xdx}{\sqrt{x+9}}$ (j) $\int_0^1 x^3(1+3x^2)^{\frac{1}{2}}dx$ (k) $\int_{-1}^1 \frac{1+x^2}{1+9x^2}dx$

   (l) $\int_1^2 \frac{3x^3+9x^2-12x+4}{9x^5-12x^4+4x^3}dx$ (m) $\int_0^3 \frac{2(x-1)}{(x^2+3)(x+1)^2}dx$

3. Evaluate the definite/indefinite integrals below:

   (a) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin(x)-\cos(x))dx$ (b) $\int_0^{\frac{\pi}{2}} \cos^3(x)\sin^2(x)dx$ (c) $\int \cos(3x)\cos(x)dx$

   (d) $\int_0^{\frac{\pi}{2}} (\sin(x)+\cos(x))^2dx$ (e) $\int (1+\cos(x))^4dx$ (f) $\int_0^{\frac{\pi}{2}} \sin^6(x)\cos^2(x)dx$

   (g) $\int_0^{\pi} \sin^5(x)dx$ (h) $\int_0^{\frac{\pi}{2}} \cos^5(x)\sin^2(x)dx$ (i) $\int_0^{\frac{\pi}{2}} \cos^6(x)dx$

   (j) $\int_0^{\frac{\pi}{2}} \sin^3(x)\cos^4(x)dx$ (k) $\int \csc^4(x)\cos(x)dx$

   (l) $\int x\sin^2(x^2)dx$ (m) $\int \frac{2dx}{\cot(x/2)+\tan(x/2)}$ (n) $\int_0^{\sqrt{2\pi}} x^3\cos^2(x^2)dx$

   (o) $\int_0^{\pi} |\sin(2x)-\sin(x)|dx$ (p) $\int_0^{\pi} ||\sin(2x)|-\sin(x)|\,dx$

4. Evaluate the definite/indefinite integrals below:

   (a) $\int_0^5 \sqrt{1+3x}dx$ (b) $\int_0^3 \frac{3x+4}{\sqrt{x+1}}dx$ (c) $\int \frac{\sqrt{x}}{1+x}dx$

   (d) $\int 2(x^2+x)e^{2x}dx$ (e) $\int \left(\frac{1}{x}+\frac{1}{x^2}\right)\ln(x)dx$ (f) $\int e^{-5x}\sin(4x)dx$

   (g) $\int \frac{dx}{x^2-3x+2}$ (h) $\int \frac{x^5dx}{x^3-1}$. (Try not to 'break up' x^3-1.)

   (i) $\int \frac{x+1}{x^2(x^2+1)}dx$ (j) $\int \frac{1-2x}{x^3+x^2+x+1}dx$ (k) $\int_0^1 \frac{x^3+x}{x^3+1}dx$

   (l) $\int \frac{24dx}{x^3-x^2-9x+9}$ (m) $\int_0^{\frac{\pi}{4}} \sec^4(x)dx$ (n) $\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\cos(x))^2}$

   (o) $\int_0^{\frac{\pi}{2}} x\sin^2(x)dx$ (p) $\int (1+\sec(x))\tan(x)dx$ (q) $\int_0^{\frac{\pi}{4}} \tan^3(x)dx$

   (r) $\int_0^{\frac{\pi}{3}} (1+\tan^6(x))dx$ (s) $\int_0^{\frac{\pi}{6}} \sin(x)\tan(x)dx$ (t) $\int \cos^2(x)\cot(x)dx$

   (u) $\int \frac{dx}{(1+\cos(x))\sin(x)}$ (v) $\int_0^{\frac{\pi}{6}} \frac{\tan(x)dx}{1+\sin^2(x)}$ (w) $\int \frac{\cos^2(2x)}{\sin^4(x)\cos^2(x)}dx$

(x) $\displaystyle\int \frac{1+\cos(x)}{x+\sin(x)}dx$ (y) $\displaystyle\int \frac{x+\sin(x)}{1+\cos(x)}dx$ (z) $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\frac{1-\sin(x)}{1+\sin(x)}}dx$

5. (a) i. Express $5+4\cos(x)+\sin(x)$ in the form $a\cos^2\left(\frac{x}{2}\right)+b\sin^2\left(\frac{x}{2}\right)$, where a, b are constants.

   ii. Hence, or otherwise, compute $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+4\cos(x)+3\sin(x)}$.

   (b) Apply the above method, or otherwise, to evaluate the definite/indefinite integrals below:

   i. $\displaystyle\int \frac{dx}{1+\sin(x)+\cos(x)}$

   ii. $\displaystyle\int \frac{dx}{4\cos(x)+3\sin(x)}$

   iii. $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos(x)}$

6. (a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3+2\sin(x)+\cos(x)}$.

   (b) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{(2\sin(x)+\cos(x))dx}{3+2\sin(x)+\cos(x)}$.

7. Show that $\displaystyle\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}dx = \int_0^{\pi} \frac{(\pi-x)\sin(x)}{1+\cos^2(x)}dx$. Hence, or otherwise, evaluate both definite integrals.

8. (a) Show that $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4(x)}{\sin^4(x)+\cos^4(x)}dx$.

   (b) Hence, or otherwise, compute $\displaystyle\int_0^{\pi} \frac{\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx$.

   (c) Show that $\displaystyle\int_0^{\pi} \frac{x\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx = \int_0^{\pi} \frac{(\pi-x)\sin^4(x)}{\sin^4(x)+\cos^4(x)}dx$. Hence, or otherwise, evaluate both definite integrals.

9. Let n be a positive integer.

   (a) Show that $\cos(x)+\cos(3x)+\cos(5x)+\cdots+\cos((2n-1)x) = \dfrac{\sin(2nx)}{2\sin(x)}$ whenever $\sin(x)\neq 0$.

   (b) Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(2nx)}{\sin(x)}dx$

   (c) Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\sin(x)+3\sin(3x)+5\sin(5x)+\cdots+(2n-1)\sin((2n-1)x))\,dx$.

10. For any positive integer n, define $I_n = \displaystyle\int_0^1 x^n\sqrt{1-x}dx$.

   (a) Show that whenever $n\geq 2$, $I_n = \left(\dfrac{2n}{2n-3}\right)I_{n-1}$.

   (b) Evaluate I_{10}.

11. For any positive integers m, n, define $I_{m,n} = \displaystyle\int_0^{\frac{\pi}{2}} \sin^m(x)\cos^n(x)dx$. Show that, whenever $m, n\geq 2$,

$$I_{m,n} = \left(\frac{m-1}{m+n}\right)I_{m-2,n} = \left(\frac{n-1}{m+n}\right)I_{m,n-2} = \left(\frac{n-1}{m+1}\right)I_{m+2,n-2} = \left(\frac{m-1}{n+1}\right)I_{m-2,n+2}.$$

12. (a) Let n be a non-negative integer. Show that the function $(-1)^n e^x \frac{d^n}{dx^n}(x^n e^{-x})$ on $\mathbb{R}$ is a polynomial function of degree n with leading coefficient 1. (Hint: Apply Leibniz's Rule.)

(b) For each non-negative integer, define the function $L_n : \mathbb{R} \longrightarrow \mathbb{R}$ by $L_n(x) = (-1)^n e^x \frac{d^n}{dx^n}(x^n e^{-x})$.

Suppose m, n are positive integers.

i. Show that

$$\lim_{x\to+\infty} \int_0^x L_n(t)L_m(t)e^{-t}dt = (-1)^{n-1} \lim_{x\to+\infty} \int_0^x \frac{d^{n-1}}{dt^{n-1}}\left(t^n e^{-t}\right) L'_m(t)dt,$$

ii. Hence, or otherwise, show that

$$\lim_{x\to+\infty} \int_0^x L_n(t)L_m(t)e^{-t}dt = \begin{cases} (n!)^2 & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m \end{cases}$$

13. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Suppose f is differentiable on $\mathbb{R}$ and $f(0) = f(1) = 0$.

Further suppose $\int_0^1 (f(x))^2 dx = 1$. Show that $\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$.

14. Evaluate the limits below.

(a) $\lim_{h\to 0} \frac{1}{h} \int_0^{\sin(h)} \sin(\sqrt{t^2 + t^4})dt$ (b) $\lim_{h\to 0} \frac{1}{h\sin(h)} \int_0^{h^2} e^{t^2} dt$ (c) $\lim_{h\to 0} \frac{1}{h} \int_{-h}^{h} \left|\sqrt[3]{\sin^5(t)}\right| dt$

(d) $\lim_{h\to 0^+} \frac{1}{\ln(1+h)} \int_2^{3h+2} \sqrt{t^6 + 2t^4 + 3t^2 + 4}dt$

15. Define $f : (0, +\infty) \longrightarrow \mathbb{R}$ by $f(x) = \int_{x^{-1}}^{x} \cos(\sqrt{xt})dt$ for any $x \in (0, +\infty)$.

(a) Show that $f(x) = \frac{1}{x}\int_1^{x^2} \cos(\sqrt{u})du$ for any $x \in (0, +\infty)$.

(b) Find the value of $f'(1)$.

16. Evaluate the first derivative of the functions of x below.

(a) $\int_{-2}^{x^3} x\sqrt{t^4 + t + 1}dt.$ (b) $\int_x^{2x} e^{3t^2} dt$ (c) $\int_{-x}^{x} |\cos(t)|^{\frac{7}{2}} dt$ (d) $\int_0^{\sin(x)} \frac{\cos^2(t^2)}{2 + t^2} dt$

(e) $\int_0^{\sin(x)} \frac{\cos(x^2)\cos(t^2)}{2 + t} dt$ (f) $\int_x^{x^2} \sin\left(\frac{t}{x^2}\right) dt$ (g) $\int_{20}^{x} \left(\int_{10}^{u} \frac{dt}{1 + t^4 + \sin^4(t)}\right) du$

17. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[0, 1]$. Further suppose that $\int_0^x f(t)dt = \int_x^1 f(t)dt$ for any $x \in [0, 1]$. Show that $f(x) = 0$ for any $x \in [0, 1]$.