THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018 Term 2

Midterm Examination

Name: ____

Student ID: ________________ Programme: ____________ Section: MATH1010____

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INSTRUCTIONS to students:

1. Answer all questions. <u>Show work to justify all answers.</u>
2. The examination lasts 120 minutes.
3. There are a total of 80 points.
4. Answer the questions in the space provided.

FOR MARKERS' USE ONLY:

1	
2	
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Total	/80 points

1. (10 marks) Let a_n be a sequence defined by $a_1 = 4$ and $a_{n+1} = 4 - \dfrac{1}{a_n}$ for $n \geq 1$.

   (a) Prove that $a_n > 3$ for any $n \geq 1$.

   (b) Prove that $\lim\limits_{n\to\infty} a_n$ exists.

   (c) Find $\lim\limits_{n\to\infty} a_n$.

Solution.

(a) Let $P(n)$ be the statement "$a_n > 3$". Clearly $P(1)$ is true. Assume $P(n)$ is true, then $a_{n+1} = 4 - \frac{1}{a_n} > 4 - \frac{1}{3} > 3$, so $P(n+1)$ is true. By the principle of mathematical induction, $a_n > 3$ for any $n \geq 1$

(b) We claim that the sequence is decreasing. Let $P(n)$ be the statement "$a_{n+1} < a_n$". Since $a_2 = \frac{15}{4} < 4 = a_1$, $P(1)$ is true. Assume $P(n)$ is true, then $a_{n+2} = 4 - \frac{1}{a_{n+1}} < 4 - \frac{1}{a_n} = a_{n+1}$, so $P(n+1)$ is true. By the principle of mathematical induction, the sequence is decreasing.

Hence, by the monotone convergence theorem, the sequence converges.

(c) Let $L = \lim\limits_{n\to\infty} a_n$. Let $n \to \infty$ in $a_{n+1} = 4 - \dfrac{1}{a_n}$, we get $L = 4 - \frac{1}{L}$. By part (a) we have $L > 3$, so $L = 2 + \sqrt{3}$

2. (6 marks)

(a) Find $\lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}}$.

(b) Find $\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2+1}} + \frac{1}{\sqrt{4n^2+2}} + \cdots + \frac{1}{\sqrt{4n^2+n}} \right)$.

Solution.

(a) $\lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{4+1/n}} = \frac{1}{2}$

(b) Note that

$$\frac{n}{\sqrt{4n^2+1}} > \left(\frac{1}{\sqrt{4n^2+1}} + \frac{1}{\sqrt{4n^2+2}} + \cdots + \frac{1}{\sqrt{4n^2+n}} \right) > \frac{n}{\sqrt{4n^2+n}}$$

, and

$$\lim_{n\to\infty} \frac{n}{\sqrt{4n^2+1}} = \frac{1}{2}$$

Together with the result in (a), we get that

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2+1}} + \frac{1}{\sqrt{4n^2+2}} + \cdots + \frac{1}{\sqrt{4n^2+n}} \right) = \frac{1}{2}$$

3. (16 marks) Evaluate the following limits.

(a) $\lim\limits_{x\to 0} \dfrac{(e^{5x}-e^{2x})^2}{x\sin x}$

$= \lim\limits_{x\to 0} \dfrac{e^{4x}(e^{3x}-1)^2}{x\sin x}$

$= \lim\limits_{x\to 0} 9\cdot e^{4x}\cdot(\dfrac{e^{3x}-1}{3x})^2\cdot\dfrac{x}{\sin x}$

$= 9\cdot\lim\limits_{x\to 0} e^{4x}\cdot(\lim\limits_{x\to 0}\dfrac{e^{3x}-1}{3x})^2\cdot\lim\limits_{x\to 0}\dfrac{x}{\sin x}$

$= 9\cdot 1\cdot(1)^2\cdot 1$

$= 9$

(b) $\lim\limits_{x\to 0} \dfrac{\sin^3 4x}{x^2\ln(1+3x)}$

$= 16\cdot(\lim\limits_{x\to 0}\dfrac{\sin 4x}{4x})^2\cdot\lim\limits_{x\to 0}\dfrac{\sin 4x}{\ln(1+3x)}$

$= 16\cdot 1\cdot\lim\limits_{x\to 0}\dfrac{4\cos 4x}{\frac{3}{1+3x}}$

$= \dfrac{64}{3}$

(c) $\lim\limits_{x\to+\infty}(x^2-\sqrt{x^4-8x^2+3})$

$= \lim\limits_{x\to+\infty}\dfrac{8x^2-3}{x^2+\sqrt{x^4-8x^2+3}}$

$= \lim\limits_{x\to+\infty}\dfrac{8-3\frac{1}{x^2}}{1+\sqrt{1-8\frac{1}{x^2}+3\frac{1}{x^4}}}$

$= 4$

(d) $\lim\limits_{x\to+\infty}\dfrac{\ln(3+\sin^2 x)}{1+x^2}$

Since $\dfrac{\ln 3}{1+x^2}\le\dfrac{\ln(3+\sin^2 x)}{1+x^2}\le\dfrac{\ln 4}{1+x^2}$

and $\lim\limits_{x\to+\infty}\dfrac{\ln 3}{1+x^2}=0, \lim\limits_{x\to+\infty}\dfrac{\ln 4}{1+x^2}=0$

By Sandwich Theorem, $\lim\limits_{x\to+\infty}\dfrac{\ln(3+\sin^2 x)}{1+x^2}=0$

4. (10 marks) Let

$$f(x) = \begin{cases} x^4 \cos\left(\dfrac{1}{x^2}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) Find $f'(0)$.

(c) Determine whether $f'(x)$ is continuous at $x = 0$.

Solution.

(a) Find $f'(x)$ for $x \neq 0$.

(b) Find $f'(0)$.

(c) Determine whether $f'(x)$ is continuous at $x = 0$.

Solution.

(a) For $x \neq 0$, $f'(x) = 4x^3 \cos(\frac{1}{x^2}) + 2x \sin(\frac{1}{x^2})$

(b)

$$f'(0) = \lim_{t \to 0} \frac{t^4 \cos\left(\dfrac{1}{t^2}\right) - f(0)}{t} = 0$$

, since $\lim_{t \to 0} t^3 = 0$ and $-1 \leq \cos\left(\dfrac{1}{t^2}\right) \leq 1$.

(c) Compute that

$$\lim_{x \to 0} f'(x) = \lim_{x \to 0} (4x^3 \cos(\frac{1}{x^2}) + 2x \sin(\frac{1}{x^2})) = 0 = f'(0)$$

This really means $f'(x)$ is continuous at $x = 0$

5. (16 marks) Find $\frac{dy}{dx}$ of the following.

(a) $y = \sqrt{x}e^{5x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}e^{5x} + 5\sqrt{x}e^{5x}$$

(b) $y = \tan\left(\frac{x}{\sqrt{1+x^2}}\right)$

$$\frac{dy}{dx} = \sec^2(\frac{x}{\sqrt{1+x^2}}) \cdot (\frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{1+x^2})$$

(c) $xy^3 + \cos(xy) = 2$

Differentiating by x on both sides,

$y^3 + 3xy^2\frac{dy}{dx} - \sin(xy)(y + x\frac{dy}{dx}) = 0$

$\frac{dy}{dx} = \frac{y\sin(xy) - y^3}{3xy^2 - x\sin(xy)}$

(d) $y = x^{(\ln x)^2}$

Taking logarithm on both sides,

$\ln y = (\ln x)^2 \ln x$

Differentiating by x on both sides,

$\frac{1}{y}\frac{dy}{dx} = \frac{3(\ln x)^2}{x}$

$\frac{dy}{dx} = \frac{3(\ln x)^2 y}{x} = \frac{3(\ln x)^2 x^{(\ln x)^2}}{x}$

6. (10 marks) Let $a \in \mathbb{R}$ and $c > 0$. Let $f(x)$ be a function which is continuous on $[a-c, a+c]$ and $f''(x) > 0$ for any $x \in (a-c, a+c)$.

(a) By applying mean value theorem, prove that there exists $p_1 \in (a-c, a)$ such that

$$f'(p_1) = \frac{f(a) - f(a-c)}{c}.$$

(b) Prove that there exists $\xi \in (a-c, a+c)$ such that

$$f(a) > \frac{f(a-c) + f(a+c)}{2} - c^2 f''(\xi).$$

Solution.

(a) By applying MVT to $f(x)$ on $[a-c, a]$, we get the desired result.

(b) Note that to prove the inequality amounts to to prove the following inequality

$$f''(\xi) > \frac{f(a+c) - f(a)}{2c^2} - \frac{f(a) - f(a-c)}{2c^2}$$

We first apply MVT to f to get $p_1 \in (a-c, a)$ and $p_2 \in (a, a+c)$ such that $f'(p_1) = \dfrac{f(a) - f(a-c)}{c}$ and $f'(p_2) = \dfrac{f(a+c) - f(a)}{c}$.

Then the L.H.S. of the latter inequality $= \dfrac{f'(p_2) - f'(p_1)}{2c}$

On the other hand, by MVT again, there exists $\xi \in (p_1, p_2)$ such that $f''(\xi) = \dfrac{f'(p_2) - f'(p_1)}{p_2 - p_1}$.

Since $f''(x) > 0$ for $x \in (a-c, a+c)$, $f'(p_2) - f'(p_1) > 0$. And since $p_2 - p_1 < 2c$, we have

$$f''(\xi) = \frac{f'(p_2) - f'(p_1)}{p_2 - p_1} > \frac{f'(p_2) - f'(p_1)}{2c}$$

i.e.

$$f''(\xi) > \frac{f(a+c) - f(a)}{2c^2} - \frac{f(a) - f(a-c)}{2c^2}$$

7. (12 marks) Let $a_1, a_2, \ldots, a_n > 0$ be positive real numbers.

(a) Prove that $x \leq e^{x-1}$ for any real number $x \in \mathbb{R}$.

(b) Using (a) or otherwise, prove that for any $\alpha > 0$,

$$a_1 a_2 \leq \alpha^2 e^{\frac{a_1+a_2}{\alpha}-2}.$$

(c) Prove that

$$a_1 a_2 \cdots a_n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n.$$

Solution.

(a) When $x = 1$, LHS = RHS.
$\frac{d(x)}{dx} = 1, \frac{d(e^{x-1})}{dx} = e^{x-1}$
For $x < 1$, $1 > e^{x-1}$, so $x \leq e^{x-1}$ for $x < 1$.
For $x > 1$, $1 < e^{x-1}$, so $x \leq e^{x-1}$ for $x > 1$.
Therefore, $x \leq e^{x-1}$ for any real number $x \in \mathbb{R}$.

(b) Put $x = \frac{a_1}{\alpha}$ in (a), we get $\frac{a_1}{\alpha} \leq e^{\frac{a_1}{\alpha}-1}$
Put $x = \frac{a_2}{\alpha}$ in (a), we get $\frac{a_2}{\alpha} \leq e^{\frac{a_2}{\alpha}-1}$
Multiplying the two inequalities results in $a_1 a_2 \leq \alpha^2 e^{\frac{a_1+a_2}{\alpha}-2}$.

(c) Following similar steps in (b), for $i = 1, 2, \cdots, n$,
Put $x = \frac{a_i}{\alpha}$ in (a), we get $\frac{a_i}{\alpha} \leq e^{\frac{a_i}{\alpha}-1}$
Multiplying all n inequalities results in $a_1 a_2 \cdots a_n \leq \alpha^n e^{\frac{a_1+a_2+\cdots+a_n}{\alpha}-n}$.

Put $\alpha = \frac{a_1 + a_2 + \cdots + a_n}{n}$ in the above inequality, we get
$a_1 a_2 \cdots a_n \leq (\frac{a_1 + a_2 + \cdots + a_n}{n})^n e^{n-n}$
Hence, $a_1 a_2 \cdots a_n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n$.