THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 9
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Exercise 1:

Evaluate the following integrals.

(a) $\displaystyle\int \frac{x}{\sqrt{1+x^2}}dx$

(b) $\displaystyle\int e^x \cos e^x dx$

(c) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx$

Exercise 2:

Evaluate the following integrals.

(a) $\displaystyle\int \sin^2 x dx$

(b) $\displaystyle\int \csc x dx \left(\text{Hint: consider } \csc x = \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} \right)$

(c) $\displaystyle\int \frac{1}{\sin^2 x \cos^2 x}dx$

Exercise 3:

Suppose $t = \tan \dfrac{\theta}{2}$.

(a) Show that

$$d\theta = \frac{2}{1+t^2}dt$$

(b) Show that

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

(c) By the fact that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2},$$

evaluate the integral

$$\int \frac{1}{5+4\cos\theta}d\theta$$

Exercise 4:

Find $\displaystyle\int f(x)$, where

$$f(x) = \begin{cases} (x+1)^3 & x > 0 \\ (x+1)^2 & x < 0 \\ 0 & x = 0 \end{cases}$$

Solution

Exercise 1:

(a)

$$\int \frac{x}{\sqrt{1+x^2}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}dx^2 = \frac{1}{2}\int \frac{1}{\sqrt{1+x^2}}d(1+x^2) = \frac{1}{2}\int d\, 2\sqrt{1+x^2} = \sqrt{1+x^2}+C$$

(b)

$$\int e^x \cos e^x dx = \int \cos e^x de^x = \sin e^x + C$$

(c)

$$\int \frac{x^5}{(1+x^3)^3}dx = \int \frac{x^3}{(1+x^3)^3}x^2 dx = \frac{1}{3}\int \frac{x^3}{(1+x^3)^3}dx^3 = \frac{1}{3}\int \frac{(1+x^3)-1}{(1+x^3)^3}dx^3$$
$$= \frac{1}{3}\int \left(\frac{1}{(1+x^3)^2} - \frac{1}{(1+x^3)^3}\right)d(1+x^3) = -\frac{1}{3(1+x^3)} + \frac{1}{6(1+x^3)^2} + C$$

Exercise 2:

(a)

$$\int \sin^2 x dx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \int \frac{\cos 2x}{2}dx = \frac{x}{2} - \int \frac{\cos 2x}{4}d2x = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b)

$$\int \csc x dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x}dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx$$
$$= \int \frac{1}{\csc x + \cot x}d(-\cot x - \csc x) = -\ln|\cot x + \csc x| + C$$

(c)

$$\int \frac{1}{\sin^2 x \cos^2 x}dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}dx = \int \frac{1}{\cos^2 x}dx + \int \frac{1}{\sin^2 x}dx$$
$$= \int \sec^2 x dx + \int \csc^2 x dx = \tan x - \cot x + C$$

Exercise 3:

Suppose $t = \tan\frac{\theta}{2}$.

(a)

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$
$$dt = \frac{1}{2}\sec^2\frac{\theta}{2}d\theta \Longrightarrow 2\cos^2\frac{\theta}{2}dt = d\theta \Longrightarrow d\theta = \frac{2}{1+t^2}dt$$

(b)

$$\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \frac{2t}{1-t^2}$$
$$\cos\theta = \frac{1-t^2}{1+t^2}$$

(c)

$$\int \frac{2}{5+4\cos\theta}d\theta = \int \frac{2}{5+4\dfrac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt = \int \frac{2}{5(1+t^2)+4(1-t^2)}dt = \int \frac{2}{9+t^2}dt$$

Let $t = 3x$

$$\int \frac{2}{9+t^2}dt = 3\int \frac{2}{9+9x^2}dx = \frac{2}{3}\int \frac{1}{1+x^2}dx = \frac{2}{3}\tan^{-1}x + C$$

$$= \frac{2}{3}\tan^{-1}\frac{t}{3} + C = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{\theta}{2}\right) + C$$

Hence,

$$\int \frac{2}{5+4\cos\theta}d\theta = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{\theta}{2}\right) + C$$

Exercise 4:

$$F(x) = \int f(x) = \begin{cases} \frac{1}{4}(x+1)^4 + C & x > 0 \\ \frac{1}{3}(x+1)^3 + C' & x < 0 \\ C'' & x = 0 \end{cases}$$

Note that $F(x)$ is continuous.

$$\lim_{x\to 0^+} F(x) = \lim_{x\to 0^-} F(x) = F(0)$$

Hence,

$$\frac{1}{4} + C = \frac{1}{3} + C' = C''$$

Therefore,

$$F(x) = \int f(x) = \begin{cases} \frac{1}{4}(x+1)^4 + C & x > 0 \\ \frac{1}{3}(x+1)^3 - \frac{1}{12} + C & x < 0 \\ \frac{1}{4} + C & x = 0 \end{cases}$$