THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 7
CHAK Wai Ho

L'Hopital's Rule
Suppose $b > a$ and $c \in (a, b)$. Let $f, g : (a, b) \to \mathbb{R}$ be a differentiable function on $(a, b) \setminus \{c\}$.
Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0(\pm\infty)$. If $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

Taylor's Polynomial
Let f be a function that is k times differentiable on (a, b). Let $x_0 \in (a, b)$.
The k-th Taylor Polynomial centered at x_0 is

$$P_k(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + ... + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Exercise 1 (continuation of Ex2 from the last tutorial) :
Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is the continuous function defined by

$$f(x) = \begin{cases} \frac{\sin(x)}{x} & x \in [-\pi, \pi] \setminus \{0\} \\ a & x = 0 \end{cases}$$

(i) Show that f is differentiable at $x = 0$.

(ii) Determine whether f is twice differentiable at $x = 0$.

Exercise 2:

Evaluate the following limits by L'Hopital's rule.

(i) $\lim_{x\to 0} \frac{\sin 4x}{\sin 5x}$ (ii) $\lim_{x\to 0} \frac{\ln \cos 2x}{\ln \cos x}$

(iii) $\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1}$

Exercise 3:

Find the n-th Taylor polynomial of $f(x)$ centered at x_0, where

(a) $n = 10, \quad f(x) = e^{x^2}, \quad x_0 = 0$

(b) $n = 3, \quad f(x) = \sin x, \quad x_0 = \frac{\pi}{2}$

Exercise 4:

(a) Write down the Taylor polynomial $P_3(x)$, where $f(x) = \ln(1 - x)$ centered at 0.

(b) Hence, approximate $\ln 0.99$.

Solution

Exercise 1:

(i) $a = 1$. (Check the previous tutorial)

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{\sin h}{h} - 1}{h} = \lim_{h\to 0} \frac{\sin h - h}{h^2}$$

As $h \to 0$, $\sin h - h \to 0$ and $h^2 \to 0$. By L'Hopital's Rule

$$\lim_{h\to 0} \frac{\sin h - h}{h^2} = \lim_{h\to 0} \frac{\cos h - 1}{2h}$$

As $h \to 0$, $\cos h - 1 \to 0$ and $2h \to 0$. By L'Hopital's Rule

$$\lim_{h\to 0} \frac{\cos h - 1}{2h} = \lim_{h\to 0} \frac{-\sin h}{2} = 0$$

Hence $f(x)$ is differentiable on $x = 0$.

(ii) For $x \neq 0$,

$$f'(x) = \frac{x\cos x - \sin x}{x^2}$$

$$\lim_{h\to 0} \frac{f'(h) - f'(0)}{h} = \lim_{h\to 0} \frac{h\cos h - \sin h}{h^3}$$

As $h \to 0$, $h\cos h - \sin h \to 0$ and $h^3 \to 0$. By L'Hopital's Rule

$$\lim_{h\to 0} \frac{h\cos h - \sin h}{h^3} = \lim_{h\to 0} \frac{\cos h - h\sin h - \cos h}{3h^2} = -\lim_{h\to 0} - \frac{\sin h}{3h} = -\frac{1}{3}$$

Hence $f(x)$ is twice differentiable on $x = 0$.

Exercise 2:

(i) As $x \to 0$, $\sin 4x \to 0$ and $\sin 5x \to 0$. By L'Hopital's Rule

$$\lim_{x\to 0} \frac{\sin 4x}{\sin 5x} = \lim_{x\to 0} \frac{4\cos 4x}{5\cos 5x} = \frac{4}{5}$$

(ii) As $x \to 0$, $\ln\cos 2x \to 0$ and $\ln\cos x \to 0$. By L'Hopital's Rule

$$\lim_{x\to 0} \frac{\ln\cos 2x}{\ln\cos x} = \lim_{x\to 0} \frac{-2\tan 2x}{-\tan x} = \lim_{x\to 0} \frac{2\tan 2x}{\tan x}$$

As $x \to 0$, $2\tan 2x \to 0$ and $\tan x \to 0$. By L'Hopital's Rule

$$\lim_{x\to 0} \frac{2\tan 2x}{\tan x} = \lim_{x\to 0} \frac{4\sec^2 2x}{\sec^2 x} = 4$$

(iii)

$$\lim_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1} = \lim_{x\to 1} \frac{x - 1 - \ln x}{(x-1)\ln x}$$

As $x \to 1$, $x - 1 - \ln x \to 0$ and $(x-1)\ln x \to 0$. By L'Hopital's Rule

$$\lim_{x\to 1} \frac{x - 1 - \ln x}{(x-1)\ln x} = \lim_{x\to 1} \frac{1 - \frac{1}{x}}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1} \frac{x-1}{x\ln x + x - 1}$$

As $x \to 1$, $x - 1 \to 0$ and $x\ln x + x - 1 \to 0$. By L'Hopital's Rule

$$\lim_{x\to 1} \frac{x-1}{x\ln x + x - 1} = \lim_{x\to 1} \frac{1}{\ln x + 1 + 1} = \frac{1}{2}$$

Exercise 3:

(a) Note that the 5-th Taylor polynomial of e^u centered at 0 is

$$1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!}$$

Put $u = x^2$. Then the 10-th Taylor polynomial of e^{x^2} centered at 0 is

$$1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \frac{x^{10}}{5!}$$

In other words,

$$1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{x^{10}}{120}$$

(b)

$$f(x) = \sin x, \quad f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x$$

$$f\left(\frac{\pi}{2}\right) = 1, \quad f'\left(\frac{\pi}{2}\right) = 0, \quad f''\left(\frac{\pi}{2}\right) = -1, \quad f'''\left(\frac{\pi}{2}\right) = 0$$

The 3-rd Taylor polynomial of $f(x) = \sin x$ centered at $\frac{\pi}{2}$ is

$$1 - \frac{(x - \frac{\pi}{2})^2}{2!} = 1 - \frac{(x - \frac{\pi}{2})^2}{2}$$

Exercise 4:

(a)

$$f(x) = \ln(1-x), \quad f'(x) = -\frac{1}{1-x}, \quad f''(x) = -\frac{1}{(1-x)^2}, \quad f'''(x) = -\frac{2}{(1-x)^3}$$

$$f(0) = 0, \quad f'(0) = -1, \quad f''(0) = -1, \quad f'''(0) = -2$$

$$P_3(x) = -x - \frac{1}{2!}x^2 - \frac{2}{3!}x^3 = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

(b) We approximate $\ln 0.99$ by $P_3(x)$, where $x = 0.01$.

$$P_3(0.01) = -0.01 - \frac{0.01^2}{2} - \frac{0.01^3}{3} \approx -0.01005$$