THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 4
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1. Differentiable Function

1. Definition

Let $f : X \to \mathbb{R}$ be a function. The function f is said to be differentiable at $x \in X$ if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$

We denote this limit as $f'(x)$.
We say $f(x)$ is differentiable on (a, b) if f is differentiable at all points in (a, b).

Remark: $f'(x)$ is the slope of the tangent line on the graph of f at x.

2. Theorem

1. Differentiability and Continuity

Let $f : X \to \mathbb{R}$ be a function.
If $f(x)$ is differentiable at $x \in X$, then $f(x)$ is continuous at x.

2. Leinbiz's Rule

We denote $f^{(n)}(x) = \frac{d^n}{dx^n} f(x)$.

Let $f, g : X \to \mathbb{R}$ be functions. Then

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} C_k^n f^{(n-k)}(x) g^{(k)}(x)$$

2. Increasing (Decreasing) Function

1. Definition

Let $f : X \to \mathbb{R}$ be a function.
The function f is said to be monotonically increasing (decreasing) if
for any $x, y \in X$, if $x < y$, then $f(x) \leq f(y)$ ($f(x) \geq f(y)$).

The function f is said to be strictly increasing (decreasing) if
for any $x, y \in X$, if $x < y$, then $f(x) < f(y)$ ($f(x) > f(y)$).

2. Corollary

Let $f : X \to \mathbb{R}$ be a differentiable function.

1. If $f'(x) > 0$ for any $x \in X$, then f is strictly increasing.
2. $f'(x) \geq 0$ for any $x \in X$ if and only if f is monotonically increasing.

Exercise 1:

Find $\dfrac{dy}{dx}$ by using first principle.

(a) $y = \sin 3x$

(b) $y = \dfrac{1}{\ln x}$

Exercise 2:

Find $\dfrac{dy}{dx}$ without using first principle.

(a) $y = e^x \sin x$

(b) $y = \dfrac{e^{3x}}{1+x}$

(c) $y = \ln(\tan^{-1} x)$

(d) $y = (\sin x)^x$

(e) $xy^2 + \cos(x+y) = 1$

Exercise 3:

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \tan^{-1} \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(a) Find $f'(x)$ for $x \neq 0$.

(b) Determine whether f is differentiable at $x = 0$.

(c) Determine whether f' is continuous at $x = 0$.

Exercise 4:

Let $y = e^{x^2}$. Show that

(a) $y' = 2xy$

(b) $y^{(n+1)}(x) = 2xy^{(n)}(x) + 2ny^{(n-1)}(x)$

Exercise 5:

(a) Let $f : (1, \infty) \to (0, \infty)$ be the function defined by

$$f(x) = \frac{x}{\ln x}$$

Show that for all $x > 1,\ f(x) \geq e$.

(b) Let $b > 1$. Let $g : (1, \infty) \to (0, \infty)$ be the function defined by

$$g(x) = \frac{x^b}{b^x}$$

Show that

(i) g is strictly increasing on $\left(1, \dfrac{b}{\ln b}\right)$ and strictly decreasing on $\left(\dfrac{b}{\ln b}, \infty\right)$;

(ii) If $1 < a < b < e$, then $a^b < b^a$.

Solution

Exercise 1:

(a) Please verify it yourself.

(b)

$$\begin{aligned}
\lim_{h\to 0}\frac{\dfrac{1}{\ln(x+h)}-\dfrac{1}{\ln x}}{h} &= \lim_{h\to 0}\frac{1}{h}\frac{\ln x-\ln(x+h)}{\ln x\ln(x+h)}\\
&= \lim_{h\to 0}\frac{\dfrac{1}{h}\left(\ln\dfrac{x}{x+h}\right)}{\ln x\ln(x+h)}\\
&= \frac{1}{x}\lim_{h\to 0}\frac{-\dfrac{x}{h}\ln\left(1+\dfrac{h}{x}\right)}{\ln x\ln(x+h)}\\
&= \frac{1}{x}\lim_{h\to 0}\frac{-\dfrac{x}{h}\ln\left(1+\dfrac{1}{\frac{x}{h}}\right)}{\ln x\ln(x+h)}\\
&= \frac{1}{x}\lim_{h\to 0}\frac{-\ln\left(1+\dfrac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}}{\ln x\ln(x+h)}\\
&= -\frac{1}{x(\ln x)^2}\lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\\
&= -\frac{\ln e}{x(\ln x)^2}\\
&= -\frac{1}{x(\ln x)^2}
\end{aligned}$$

Exercise 2:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = e^x(\sin x+\cos x)$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(3x+2)e^{3x}}{(x+1)^2}$$

(c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2+1)\tan^{-1}x}$$

(d)

$$\begin{aligned}
\ln y &= x\ln\sin x\\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \ln\sin x + x\cot x\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= (\sin x)^x(\ln\sin x + x\cot x)
\end{aligned}$$

(e)

$$y^2+2xy\frac{\mathrm{d}y}{\mathrm{d}x}-\sin(x+y)(1+\frac{\mathrm{d}y}{\mathrm{d}x})=0$$

$$y^2 + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} - \sin(x+y) - \sin(x+y)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(x+y) - y^2}{2xy - \sin(x+y)}$$

Exercise 3:

(a) $f'(x) = 2x\tan^{-1}\frac{1}{x} - \frac{x^2}{x^2+1}$

(b) One has
$$\left|\frac{f(h)-f(0)}{h-0}\right| = \left|\frac{f(h)}{h}\right| \le \left|h\tan^{-1}\frac{1}{h}\right| \le |h|\cdot\frac{\pi}{2}$$
By squeeze theorem, since
$$\lim_{h\to 0}|h|\cdot\frac{\pi}{2} = 0,$$
we have
$$\lim_{h\to 0}\frac{f(h)-f(0)}{h-0} = 0$$
f is differentiable at $x = 0$

(c)
$$f'(x) = \begin{cases} 2x\tan^{-1}\frac{1}{x} - \frac{x^2}{x^2+1} & x \ne 0 \\ 0 & x = 0 \end{cases}$$
One has
$$\left|x\tan^{-1}\frac{1}{x}\right| \le |x|\cdot\frac{\pi}{2}$$
By squeeze theorem, since
$$\lim_{x\to 0}|x|\cdot\frac{\pi}{2} = 0,$$
we have
$$\lim_{x\to 0} 2x\tan^{-1}\frac{1}{x} = 0$$
Also,
$$\lim_{x\to 0}\frac{x^2}{x^2+1} = 0$$
Hence,
$$\lim_{x\to 0} f'(x) = 0 = f'(0)$$
Therefore, f' is continuous at $x = 0$.

Exercise 4:

(a) Please verify it yourself.

(b)
$$y^{(n+1)} = (2xy)^{(n)} = 2C_0^n x^{(0)}y^{(n)} + 2C_1^n x^{(1)}y^{(n-1)} = 2xy^{(n)} + 2ny^{(n-1)}$$

Exercise 5:

(a) One has
$$f'(x) = \frac{\ln x - 1}{(\ln x)^2}$$
If $x \ge e$, then $f'(x) \ge 0$. Hence $f(x) \ge f(e) = e$;
If $1 < x < e$, then $f'(x) \le 0$. Hence $f(x) \ge f(e) = e$.
Hence, for $x > 1$, $f(x) \ge e$.

(b)(i) One has

$$\ln g(x) = b \ln x - x \ln b$$

Differentiating both sides,

$$\frac{1}{g(x)} g'(x) = \frac{b}{x} - \ln b$$

$$g'(x) = \frac{g(x) \ln b}{x} \left(\frac{b}{\ln b} - x \right)$$

Observe that $g(x) > 0$ for all $x > 1$, and $\ln b > 0$.

If $1 < x < \dfrac{b}{\ln b}$, we have $g'(x) > 0$ (verify it). Hence g is strictly increasing on $\left(1, \dfrac{b}{\ln b}\right)$;
If $x > \dfrac{b}{\ln b}$, we have $g'(x) < 0$ (verify it). Hence g is strictly decreasing on $\left(\dfrac{b}{\ln b}, \infty\right)$.

(b)(ii) Let $1 < a < b < e$. One has

$$g(a) = \frac{a^b}{b^a}, \quad g(b) = 1.$$

Also, since $0 < \ln b < 1$, we have $1 < b < \dfrac{b}{\ln b}$.

Since $a < b$ and g is strictly increasing on $\left(1, \dfrac{b}{\ln b}\right)$, we have

$$g(a) < g(b)$$

Hence,

$$a^b < b^a$$