THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 11
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Exercise 1:

Evaluate the following integrals by using trigonometric substitution or other methods.

(a) $\displaystyle\int \frac{1}{\sqrt{x^2-16}}dx$

(b) $\displaystyle\int \frac{1}{(4x^2+1)^{\frac{3}{2}}}dx$

(c) $\displaystyle\int \frac{1}{x^2+x+1}dx$

Exercise 2:

Evaluate the following integrals by using integration by part or other methods.

(a) $\displaystyle\int x\sec^2 x dx$

(b) $\displaystyle\int \sin\ln x dx$

Exercise 3:

Let m and n be positive integers. Define

$$I_{m,n} = \int_{-1}^{1}(1+x)^m(1-x)^n dx$$

Show that

$$I_{m,n} = \frac{n}{m+1}I_{m+1,n-1}$$

Hence show that

$$I_{m,n} = \frac{n!\ m!\ 2^{m+n+1}}{(m+n+1)!}$$

Exercise 4:

Calculate the area in the first quadrant bounded by

(a)

$$y = 4x,\ \ y = x^4,\ \ x = 1$$

(b)

$$y = x^2\tan^{-1}x,\ \ x = 1$$

(c)

$$y = 4x,\ \ y = x^2$$

Solution

Exercise 1:

(a) Let $x = 4\sec\theta$.

$$\int \frac{1}{\sqrt{x^2-16}}dx = \int \frac{1}{\sqrt{16\sec^2\theta - 16}}d(4\sec\theta) = \int \frac{1}{4\tan\theta}4\sec\theta\tan\theta d\theta = \int \sec\theta d\theta$$

$$= \ln\left|\sec\theta + \tan\theta\right| + C = \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C$$

(b) Let $x = \frac{1}{2}\tan\theta$.

$$\int \frac{1}{(4x^2+1)^{\frac{3}{2}}}dx = \int \frac{1}{(\tan^2\theta+1)^{\frac{3}{2}}}d\left(\frac{1}{2}\tan\theta\right) = \int \frac{1}{\sec^3\theta}\frac{1}{2}\sec^2\theta d\theta$$

$$= \frac{1}{2}\int \cos\theta d\theta = \frac{1}{2}\sin\theta + C = \frac{x}{\sqrt{4x^2+1}} + C$$

(c) Let $x = \frac{\sqrt{3}}{2}y - \frac{1}{2}$.

$$\int \frac{1}{x^2+x+1}dx = \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}dx = \int \frac{1}{\frac{3}{4}\left(y^2+1\right)}\frac{\sqrt{3}}{2}dy = \frac{2}{\sqrt{3}}\int \frac{1}{y^2+1}dy$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}y + C = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + C$$

Exercise 2:

(a)

$$\int x\sec^2 x dx = \int x d\tan x = x\tan x - \int \tan x dx = x\tan x - \int \frac{\tan x \sec x}{\sec x}dx$$

$$= x\tan x - \int \frac{1}{\sec x}d\sec x = x\tan x - \ln\sec x + C$$

(b)

$$\int \sin\ln x dx = x\sin\ln x - \int x d\sin\ln x = x\sin\ln x - \int \cos\ln x dx$$

$$= x\sin\ln x - x\cos\ln x + \int x d\cos\ln x = x\sin\ln x - x\cos\ln x - \int \sin\ln x dx$$

Thus

$$\int \sin\ln x dx = \frac{x\sin\ln x - x\cos\ln x}{2} + C$$

Exercise 3:

$$
\begin{aligned}
I_{m,n} &= \int_{-1}^{1} (1+x)^m (1-x)^n dx \\
&= \frac{1}{m+1} \int_{-1}^{1} (1-x)^n d(1+x)^{m+1} \\
&= \frac{1}{m+1} \left[(1-x)^n (1+x)^{m+1} \right]_{-1}^{1} - \int_{-1}^{1} (1+x)^{m+1} d(1-x)^n \\
&= \frac{n}{m+1} \int_{-1}^{1} (1-x)^{n-1} (1+x)^{m+1} dx \\
&= \frac{n}{m+1} I_{m+1,n-1} \\
&= \frac{n}{m+1} \frac{n-1}{m+2} I_{m+2,n-2} \\
&= \frac{n}{m+1} \frac{n-1}{m+2} \frac{n-2}{m+3} I_{m+3,n-3} \\
&= \frac{n}{m+1} \frac{n-1}{m+2} \frac{n-2}{m+3} \cdots \frac{1}{m+n} I_{m+n,0} \\
&= \frac{n}{m+1} \frac{n-1}{m+2} \frac{n-2}{m+3} \cdots \frac{1}{m+n} \int_{-1}^{1} (1+x)^{m+n} dx \\
&= \frac{n}{m+1} \frac{n-1}{m+2} \frac{n-2}{m+3} \cdots \frac{1}{m+n} \frac{1}{m+n+1} \left[(1+x)^{m+n+1} \right]_{-1}^{1} \\
&= \frac{n}{m+1} \frac{n-1}{m+2} \frac{n-2}{m+3} \cdots \frac{1}{m+n} \frac{1}{m+n+1} 2^{m+n+1} \\
&= \frac{n!}{(m+1)\cdots(m+n+1)} 2^{m+n+1} \\
&= \frac{n!\ m!}{(m+n+1)!} 2^{m+n+1}
\end{aligned}
$$

Exercise 4:

(a) The area is

$$
\int_0^1 (4x - x^4) dx = \left[2x^2 - \frac{1}{5} x^5 \right]_0^1 = 2 - \frac{1}{5} = \frac{9}{5}
$$

(b) The area is

$$
\int_0^1 x \tan^{-1} x dx = \int_0^1 \tan^{-1} x d\ \frac{x^2}{2} = \left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x^2}{2} d \tan^{-1} x = \left[\frac{x^2}{2} \tan^{-1} x \right]_0^1 - \int_0^1 \frac{x^2}{2(x^2+1)} dx
$$

$$
= \frac{\pi}{8} - \int_0^1 \frac{x^2+1-1}{2(x^2+1)} dx = \frac{\pi}{8} - \frac{1}{2} + \int_0^1 \frac{1}{2(x^2+1)} dx = \frac{\pi}{8} - \frac{1}{2} + \left[\frac{1}{2} \tan^{-1} x \right]_0^1 = \frac{\pi}{8} - \frac{1}{2} + \frac{\pi}{8} = \frac{\pi}{4} - \frac{1}{2}
$$

(c) Solving $y = 4x$ and $y = x^2$, we have $x = 0$ or $x = 4$.
The area is

$$
\int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{1}{3} x^3 \right]_0^4 = \frac{32}{3}
$$