THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 1
CHAK Wai Ho

1. **Trigonometry**

Here are some useful trigonometric identities:

(a) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(b) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(c) $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

The product-to-sum and the sum-to-product formulae can be derived from the above identities.

(a) $2 \sin x \cos y = \sin(x - y) + \sin(x + y)$

(b) $2 \cos x \cos y = \cos(x - y) + \cos(x + y)$

(c) $2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

(d) $\sin x \pm \cos y = 2 \sin\left(\dfrac{x \pm y}{2}\right) \cos\left(\dfrac{x \mp y}{2}\right)$

(e) $\cos x + \cos y = 2 \cos\left(\dfrac{x - y}{2}\right) \cos\left(\dfrac{x + y}{2}\right)$

(f) $\cos x - \cos y = -2 \sin\left(\dfrac{x - y}{2}\right) \sin\left(\dfrac{x + y}{2}\right)$

2. **The Limit of a Sequence**

Definition

(a) **Sequence**
A sequence is a function whose domain is $\mathbb{N}$ or a subset of $\mathbb{N}$.

(b) **Bounded Sequence**
Let $\{a_n\}$ be a sequence.
The sequence $\{a_n\}$ is said to be bounded if there exists $M \in \mathbb{R}$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.

(c) **Monotonic Sequence**
Let $\{a_n\}$ be a sequence.
The sequence $\{a_n\}$ is said to be monotonically increasing (decreasing) if
for any $m < n$, we have $a_m \leq a_n$ $(a_m \geq a_n)$.
The sequence $\{a_n\}$ is monotonic if
it is either monotonically increasing or monotonically decreasing.

Theorem

(a) **Monotone Convergence Theorem**
Let $\{a_n\}$ be a sequence. If the sequence $\{a_n\}$ is bounded and monotonic, then $\lim\limits_{n\to\infty} a_n$ exists.

(b) **Squeeze Theorem**
Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that $a_n \leq b_n \leq c_n$.
If there exists $L \in \mathbb{R}$ such that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$,
then $\{b_n\}$ is convergent and $\lim\limits_{n\to\infty} b_n = L$.

Exercise 1:

Show the following identities.

(a) $\cos 3x = 4\cos^3 x - 3\cos x$ (b) $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

Exercise 2:

Let $\{a_n\}$ be a sequence. Find $\lim_{n\to\infty} a_n$ if it exists.

(a) $a_n = \dfrac{7n+3}{3n^2+6n-4}$ (b) $a_n = \dfrac{\sqrt{9n^2+7}}{2n+3}$

(c) $a_n = \cos\left(\dfrac{n\pi}{2}\right)$ (d) $a_n = \dfrac{\sin n}{n}$

(e) $a_1 = 0$, $a_n = \displaystyle\prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right)$ for $n \geq 2$

Exercise 3:

Let $\{a_n\}$ be the sequence defined as follows:

$$a_1 = 0, \qquad a_{n+1} = \frac{2}{3}a_n + 1$$

(a) Determine whether the sequence $\{a_n\}$ is bounded.

(b) Determine whether the sequence $\{a_n\}$ is monotonic.

(c) Find the limit of the sequence $\{a_n\}$ if it exists.

Exercise 4:

(a) Let $k, n \in \mathbb{N}$. For $1 \leq k \leq n$, show that $(n+1-k)\,k \geq n$. Hence, show that $\big(n!\big)^2 \geq n^n$.

(b) State whether $\lim_{n\to\infty} \big(n!\big)^{-\frac{1}{n}}$ exists.

If yes, find the limit. If not, explain why it does not exist.

Solution

You should notice that full solutions may not be provided.

The exercises without full solutions are discussed in the tutorial classes on Thursday.

Exercise 1:

(a) Please verify it yourself.

(b) One has

$$\tan 3x = \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x} + \tan x}{1 - \dfrac{2\tan x}{1-\tan^2 x}\tan x} = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$

Exercise 2:

(a) Answer: 0.

Please verify it yourself.

(b) One has

$$a_n = \frac{\frac{1}{n}\sqrt{9n^2+7}}{\frac{1}{n}(2n+3)} = \frac{\sqrt{9+\frac{7}{n^2}}}{2+\frac{3}{n}}$$

Hence, $\lim_{n\to\infty} a_n = \frac{3}{2}$.

(c) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}$ is an alternating sequence with four repeating terms: $0, -1, 0, 1$. The limit does not exist.

(d) One has $0 \le \frac{\sin n}{n} \le \frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$. By squeeze theorem, $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(e) For $n \ge 2$, one has

$$\begin{aligned} a_n &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) \\ &= \left(1 - \frac{1}{2}\right)\cancel{\left(1 + \frac{1}{2}\right)}\cancel{\left(1 - \frac{1}{3}\right)}\cancel{\left(1 + \frac{1}{3}\right)}\cdots\cancel{\left(1 - \frac{1}{n}\right)}\left(1 + \frac{1}{n}\right) \\ &= \frac{1}{2}\cdot\frac{n+1}{n} = \frac{1}{2}\cdot\left(1 + \frac{1}{n}\right) \end{aligned}$$

Hence, $\lim_{n\to\infty} a_n = \frac{1}{2}$.

Exercise 3:

(a) Let $P(n)$ be the proposition that $0 \leq a_n \leq 3$.
Since $0 \leq a_1 = 0 \leq 3$, $P(1)$ is true.
Suppose $P(k)$ is true for some $k \in \mathbb{N}$. i.e. $0 \leq a_k \leq 3$.
Then $0 \leq 1 \leq a_{k+1} = \frac{2}{3}a_k + 1 \leq \frac{2}{3} \cdot 3 + 1 = 3$. Hence, $P(k+1)$ is true.
By the first principle of mathematical induction, $P(n)$ is true for any $n \in \mathbb{N}$.
Therefore, $0 \leq a_n \leq 3$.
In particular, $|a_n| \leq 3$ and hence $\{a_n\}$ is bounded.

(b) One has $a_{n+1} - a_n = \frac{2}{3}a_n + 1 - a_n = 1 - \frac{1}{3}a_n \geq 1 - \frac{1}{3} \cdot 3 \ \Big(\text{by } (a)\Big) = 0$.
Therefore, $a_{n+1} \geq a_n$ and hence $\{a_n\}$ is monotonic.

(c) By monotone convergence theorem, since $\{a_n\}$ is bounded and monotonic, $\lim\limits_{n\to\infty} a_n$ exists.
Let $a = \lim\limits_{n\to\infty} a_n$. One has $a = \frac{2}{3}a + 1$ and hence $a = \lim\limits_{n\to\infty} a_n = 3$.

Remark: a_n is a sum of the first n terms of a geometric sequence.

Exercise 4:

(a) Observe that for any $a, b \in \mathbb{N}$, $ab + 1 \geq a + b$ (verify it).
Put $a = n + 1 - k, b = k$. Then $(n + 1 - k)\, k \geq (n + 1 - k) + k - 1 = n$.

$$\begin{aligned}(n!)^2 &= \Big(n \cdot (n-1) \cdots 2 \cdot 1\Big)\Big(n \cdot (n-1) \cdots 2 \cdot 1\Big)\\ &= \Big(n \cdot 1\Big)\Big((n-1) \cdot 2\Big) \cdots \Big(2 \cdot (n-1)\Big)\Big(1 \cdot n\Big) \geq \underbrace{n \cdot n \cdots n}_{n \text{ times}} = n^n\end{aligned}$$

(b) By (a), one has $0 \leq \big(n!\big)^{-\frac{1}{n}} \leq n^{-\frac{1}{2}}$ and $\lim\limits_{n\to\infty} n^{-\frac{1}{2}} = 0$.
By Squeeze theorem, $\lim\limits_{n\to\infty} \big(n!\big)^{-\frac{1}{n}} = 0$.