THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H University Mathematics 2016-2017
Suggested Solution to Assignment 2

1. (a)

$$\lim_{n\to\infty}\frac{3^n-1}{3^n+1}=\lim_{n\to\infty}(1-\frac{2}{3^n+1})=1$$

(b) This sequence is not convergent.

(c)

$$\lim_{n\to\infty}(\sqrt{n+5}-\sqrt{n})=\lim_{n\to\infty}(\frac{5}{\sqrt{n+5}+5\sqrt{n}}=0$$

(d) This sequence is not convergent.

(e)

$$\lim_{n\to\infty}(\frac{3n^2}{n+1}-3n)=\lim_{n\to\infty}\frac{3n^2-3n^2-3n}{n+1}=\lim_{n\to\infty}(\frac{-3n-3}{n+1}+\frac{3}{n+1})=\lim_{n\to\infty}(-3+\frac{3}{n+1})=-3.$$

(f)

$$\lim_{n\to\infty}(2-\frac{1}{2^n})(3+\frac{2}{n^2})=[\lim_{n\to\infty}(2-\frac{1}{2^n})][\lim_{n\to\infty}(3+\frac{2}{n^2})]=(2)(3)=6$$

(g) Applying the formula $(a-b)(a^2+ab+b^2)=a^3-b^3$, letting $a=\sqrt[3]{n^2+1}, b=\sqrt[3]{n^2}$, we have

$$(\sqrt[3]{n^2+1}-\sqrt[3]{n^2})((\sqrt[3]{n^2+1})^2+(\sqrt[3]{n^2+1})(\sqrt[3]{n^2})+(\sqrt[3]{n^2})^2)=1$$

$$\sqrt[3]{n^2+1}-\sqrt[3]{n^2}=\frac{1}{(\sqrt[3]{n^2+1})^2+(\sqrt[3]{n^2+1})(\sqrt[3]{n^2})+(\sqrt[3]{n^2})^2}$$

Since $(\sqrt[3]{n^2+1})^2+(\sqrt[3]{n^2+1})(\sqrt[3]{n^2})+(\sqrt[3]{n^2})^2$ tends to infinity as n tends to infinity,

$$\lim_{n\to\infty}(\sqrt[3]{n^2+1}-\sqrt[3]{n^2})=0$$

(h) $\lim_{n\to\infty}[(1-\frac{1}{2^2})(1-\frac{1}{3^2})...(1-\frac{1}{n^2})]$
$=\lim_{n\to\infty}[(1-\frac{1}{2})(1+\frac{1}{2})(1-\frac{1}{3})(1+\frac{1}{3})...(1-\frac{1}{n})(1+\frac{1}{n})]$
$=\lim_{n\to\infty}[(\frac{1}{2})(\frac{2}{3})...(\frac{n-1}{n})][(\frac{3}{2})(\frac{4}{3})...(\frac{n+1}{n})]$
$=\lim_{n\to\infty}(\frac{1}{n})(\frac{n+1}{2})$
$=\frac{1}{2}$

2. (a)

$$\because 3-\frac{1}{2n}\le\frac{6n+\cos n}{2n}\le 3+\frac{1}{2n}\text{ and }\lim_{n\to\infty}\frac{1}{2n}=0,$$

$$\therefore\lim_{n\to\infty}\frac{6n+\cos n}{2n}=3.$$

(b)

$$\because\frac{2n^2-n}{n^2}\le\frac{2n^2+(-1)^n n}{n^2}\le\frac{2n^2+n}{n^2}\text{ and }\lim_{n\to\infty}\frac{2n^2\pm n}{n^2}=2,$$

$$\therefore\lim_{n\to\infty}\frac{2n^2+(-1)^n n}{n^2}=2.$$

3.

$$\sum_{r=n}^{2n} \frac{1}{(2n)^2} \leq \sum_{r=n}^{2n} \frac{1}{r^2} = \frac{1}{n^2} + \cdots + \frac{1}{(2n)^2} \leq \sum_{r=n}^{2n} \frac{1}{n^2},$$

and by

$$\sum_{r=n}^{2n} \frac{1}{(2n)^2} = \frac{2n-n+1}{(2n)^2} \to 0, \sum_{r=n}^{2n} \frac{1}{n^2} = \frac{2n-n+1}{n^2} \to 0,$$

thus we have

$$\lim_{n\to\infty} \frac{1}{n^2} + \cdots + \frac{1}{(2n)^2} = 0.$$

4. (a) Assume that

$$\frac{5x-3}{x(x+1)(x+3)} = \frac{C_1}{x} + \frac{C_2}{x+1} + \frac{C_3}{x+3} = \frac{(C_1+C_2+C_3)x^2 + (4C_1+3C_2+C_3)x + 3C_1}{x(x+1)(x+3)},$$

thus the following should hold:

$$\begin{aligned} &C_1 + C_2 + C_3 = 0, \\ &4C_1 + 3C_2 + C_3 = 5, \\ &3C_1 = -3. \end{aligned}$$

thus

$$C_1 = -1, C_2 = 4, C_3 = -3, \ i.e. \ \frac{5x-3}{x(x+1)(x+3)} = -\frac{1}{x} + \frac{4}{x+1} - \frac{3}{x+3}.$$

(b) By (a),

$$\sum_{k=1}^{n} \frac{5k-3}{k(k+1)(k+3)} = \sum_{k=1}^{n} \Big(-\frac{1}{k} + \frac{4}{k+1} - \frac{3}{k+3} \Big) = -\sum_{k=1}^{n} \frac{1}{k} + 4\sum_{k=1}^{n} \frac{1}{k+1} - 3\sum_{k=1}^{n} \frac{1}{k+3}$$

$$= -\sum_{k=1}^{n} \frac{1}{k} + 4\sum_{k=2}^{n+1} \frac{1}{k} - 3\sum_{k=4}^{n+3} \frac{1}{k} = -\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + 4\Big(\frac{1}{2} + \frac{1}{3} + \frac{1}{n+1}\Big) - 3\Big(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3}\Big)$$

$$\longrightarrow -\frac{1}{1} - \frac{1}{2} - \frac{1}{3} + 4\Big(\frac{1}{2} + \frac{1}{3}\Big) = \frac{3}{2}.$$

5. (a) We use the mathematical induction method. As

$$2a_1 = 0 = 2(1) - 1 + (-1)^1$$

and we assume that $2a_n = 2n - 1 + (-1)^n$,

then $2a_{n+1} = 2(2n - a_n) = 4n - 2n + 2 - 2(-1)^n = 2(n+1) - 1 + (-1)^{n+1}$.

(b)

$$\frac{a_n}{n} = 1 + \frac{-1 + (-1)^n}{2n}, \implies \lim_{n\to\infty} \frac{a_n}{n} = 1.$$

6. (a) It is obvious that it holds for $n = 1$. Using the mathematical induction, we assume that it holds for $k = 1, 2, \cdots, n$, then

$$2a_{n+1} = 4n - 2a_n = 4n - 2n + 1 - (-1)^n = 2n + 1 + (-1)^{n+1} = 2(n+1) - 1 + (-1)^{n+1}.$$

(b) By (a),

$$\frac{a_n}{n} = 1 - \frac{1}{2n} + \frac{(-1)^n}{2n} \implies 1 - \frac{1}{2n} + \frac{-1}{2n} \le \frac{a_n}{n} \le 1 - \frac{1}{2n} + \frac{1}{2n},$$

By Sandwich Theorem,

$$\lim_{n\to\infty} \frac{a_n}{n} = 0.$$

7. (a) It is obvious that it holds for $n = 2$. Using the mathematical induction, we assume that it holds for $k = 2, 3, \cdots, n$, then

$$\frac{2^{n+1}}{(n+1)!} = (\frac{2^n}{n!})(\frac{2}{n+1}) \le (\frac{4}{n})(\frac{2}{n+1}) \le \frac{4}{n+1}$$

(b) By Sandwich Theorem,

$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$

8. (a) i. Using the mathematical induction, as $x_1 = 2 > 1$, and we assume $x_k > k$, for $k \le n$, then

$$x_{n+1} = x_n^2 - x_n + 1 = (x_n - \frac{1}{2})^2 + \frac{3}{4} > (n - \frac{1}{2})^2 + \frac{3}{4} = n^2 - n + 1.$$

As $n \ge 2, n^2 - n + 1 \ge n + 1$, thus $x_{n+1} > n + 1$.

ii. Also using the indution method:

$$s_1 = \frac{1}{x_1} = \frac{1}{2}, \text{ and } 1 - \frac{1}{x_2 - 1} = 1 - \frac{1}{3-1} = \frac{1}{2},$$

thus it holds for $k = 1$, then we assume that it holds for $k \le n$.

$$s_{n+1} = s_n + \frac{1}{x_{n+1}} = 1 - \frac{1}{x_{n+1} - 1} + \frac{1}{x_{n+1}} = 1 - \frac{1}{x_{n+1}(x_{n+1} - 1)} = 1 - \frac{1}{x_{n+2} - 1}.$$

(b) As s_n is monotomic increasing, and $s_n \le 1$, thus $\lim_{n\to\infty} s_n$ exists and ≤ 1.

9. (a) For $n \ge 1$, to prove $x_{n+1} \ge y_{n+1}$, it is equivalent to prove

$$\frac{x_n + y_n}{2} \ge \frac{2x_n y_n}{x_n + y_n} \iff (x_n + y_n)^2 \ge 4x_n y_n \iff (x_n - y_n)^2 \ge 0,$$

which obviously holds.

(b)

$$y_{n+1} = \frac{2x_n y_n}{x_n + y_n} = \frac{2y_n}{1 + y_n/x_n} \ge \frac{2y_n}{1+1} = y_n, \text{ by (a): } y_n/x_n \le 1.$$

which implies y_n increasing.

Then taking $x_n + y_n = 2x_{n+1}$ into $y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$, we have that

$$\frac{x_n}{x_{n+1}} = \frac{y_{n+1}}{y_n},$$

by y_n increasing, we have that x_n decreasing.

(c) As $x_n \ge 0$ and x_n decreasing, we have that $\lim_{n\to\infty} x_n$ exists, which is denoted by x.

While $x_n - y_n \ge 0$ and is decreasing, thus $\lim_{n\to\infty}(x_n - y_n)$ exists, denoted by $z \ge 0$. Thus, $y_n = x_n - (x_n - y_n)$ has limit $y = x - z$.

Lettint $n \to \infty$ in $x_{n+1} = \frac{x_n + y_n}{2}$, we have that $x = \frac{x+y}{2}, i.e. x = y$.

(d) In proof of (b), we see that $x_n y_n = x_{n+1} y_{n+1}, \forall n$. Thus

$$xy = \lim_{n\to\infty} x_n y_n = x_1 y_1,$$

then we have that

$$x = y = \sqrt{x_1 y_1}.$$