§5.4.

Th 1 (Non-unif Th). Let $f: D \to \mathbb{R}$. Then TFSA:

(i) f is not unif cts;

(ii) $\exists\, \varepsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, u, x \in D$ with $|u - x| < \delta$ but $|f(u) - f(x)| \geq \varepsilon$;

(iii) $\exists\, \varepsilon > 0$ ~~and~~ and $\exists$ seq $(u_n), (x_n)$ in D with each $|u_n - x_n| < \frac{1}{n}$ but $|f(u_n) - f(x_n)| \geq \varepsilon$

(iii*) Same as (iii) but $|u_n - x_n| < 1/n \ \forall n$ replaced by $|u_n - x_n| \to 0$ as $n \to \infty$.

Pf. (iii) $\Rightarrow$ (i). Take ε and $(u_n), (x_n)$ as in (iii).

Let $\delta > 0$. Then $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$, & so

$$|u_n - x_n| < \frac{1}{n} < \delta$$

while $|f(u_n) - f(x_n)| \geq \varepsilon$. Therefore (i) holds.

<u>Examples</u> (a) $f(x) = \frac{1}{x}$ $\forall\, x \in (0, 1]$. Let $\varepsilon := 1$ and let $\delta > 0$. By Archimedean property, $\exists\, n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$. Define

$$u_n = \frac{1}{n} \quad \text{and} \quad x_n = \frac{1}{2n} \qquad \left(\text{both in } (0, 1]\right)$$

Then $|u_n - x_n| = \frac{1}{2n} < \delta$ but

$$|f(u_n) - f(x_n)| = n \geq 1 = \varepsilon$$

showing that f is not unif. cts on $(0, 1]$.

(b) $f(x) = x^2$ $\forall\, x \in [0, +\infty)$.

Let $\varepsilon :\triangleq 1$ and $\delta > 0$. Let $n \in \mathbb{N}$ be s.t. $\frac{1}{n} < \delta$, let $x \triangleq n$ and $u \triangleq n + \frac{1}{n}$. Then $x, u \in [0, +\infty)$ with $|x - u| = \frac{1}{n} < \delta$ but

$$|f(u) - f(x)| = \left|\left(n + \tfrac{1}{n}\right)^2 - n^2\right| > 2 > \varepsilon.$$

§5.4. Unif cts functions (Continued)

Th2 (Uniform Cts Th). Let $f: [a,b] \to \mathbb{R}$ with $-\infty < a < b < +\infty$. Then $\Leftrightarrow$ (TFSA):

(i) f is cts on $[a,b]$;

(ii) f is unif cts on $[a,b]$.

proof. (already done in an earlier Notes)

Example (on possible extensions). Let $f(x) = \frac{1}{1+x^2}$ $\forall x \in \mathbb{R}$.

First solution. Let $\varepsilon > 0$. Take $\delta \triangleq \varepsilon/2$. Then, $\forall |u-x| < \delta$, one has $|f(u)-f(x)| < \varepsilon$ because

$$|f(u)-f(x)| = \left|\frac{x^2-u^2}{(1+u^2)(1+x^2)}\right| \leq |x-u|\left(\frac{|x|}{(1+x^2)(1+u^2)} + \frac{|u|}{(1+u^2)(1+x^2)}\right)$$
$$\leq |x-u|(1+1) < 2\delta = \varepsilon$$

1st (generalization theo). Let $f: D \to \mathbb{R}$ be Lipschitz in the sense that $\exists\, k > 0$ s.t. $|f(u)-f(x)| \leq k|u-x|$ $\forall u, x \in D$. Then f is unif cts on D ($k = 2$ in the above example)

Second solution. Let $\varepsilon > 0$. Take $a < -1$ and $b > 1$ s.t.

$$|f(x) - 0| < \varepsilon/4 \quad \forall x \in (-\infty, a] \cup [b, \infty)$$

and so in particular,

① $|f(x_1) - f(x_2)| < \varepsilon/2$ $\quad \forall x_1, x_2 \in (-\infty, a]$

&

② " " $\quad \forall x_1, x_2 \in [b, +\infty)$.

(the last two remain to be valid when f has the property that $\lim\limits_{x\to-\infty} f(x) = \ell_1 \in \mathbb{R}$ & $\lim\limits_{x\to+\infty} f(x) = \ell_2 \in \mathbb{R}$)

By the Unif Cts Th (applied to $[a,b]$), $\exists\, \delta > 0$ (with loss of gen. $\delta < (b-a)$) s.t.

(3) $|f(x_1) - f(x_2)| < \frac{\varepsilon}{2}$ $\forall\, x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta$.

Then it remains to show that

$$|f(u) - f(x)| < \varepsilon \quad \forall\, u, x \in \mathbb{R} \text{ with } |u - x| < \delta.$$

To do this, let $u, x \in \mathbb{R}$ ~~be~~ with $|u - x| < \delta$, & $u \neq x$, say $u < x$. By (3), we may assume that $[a,b]$ does not contain the set $\{u, x\}$: either $u \notin [a,b]$ or $x \notin [a,b]$, that is $u < a$ or $b < x$, say

$$u < a$$

(then $x = (x-u) + u < \delta + a < (b-a) + a = b$). By (1) & (3) one has

$$|f(u) - f(a)| < \frac{\varepsilon}{2} \quad (u, a \in (-\infty, a])$$

and

$$|f(a) - f(x)| < \frac{\varepsilon}{2} \quad (\because |x - a| < \delta \text{ \& } x, a \in [a,b])$$

Consequently $|f(u) - f(x)| < \varepsilon$, as was required to show. This completes the proof for the unif cts for the function $\frac{1}{1+x^2}$, as well as for the following

Th. Let $f: \mathbb{R} \to \mathbb{R}$ be of the property that $\forall\, \varepsilon > 0$, $\exists$ reals $a < b$ s.t. $f|_{[a,b]}$ is cts and

$$|f(x_1) - f(x_2)| < \varepsilon/2 \quad \forall\, x_1, x_2 \in (-\infty, a] \quad (1)$$

&

$$|f(x_1) - f(x_2)| < \varepsilon/2 \quad \forall\, x_1, x_2 \in [a, +\infty). \quad (2)$$

Or, more generally, $\forall \varepsilon > 0$, $\exists$ reals $a < b$ and $\delta > 0$ s.t. (1) and (2) hold, and (4)

$|f(x_1) - f(x_2)| < \varepsilon/2 \ \forall \{x_1, x_2\} \subseteq [a, b]$ with $|x_1 - x_2| < \delta$ (3)

Then f is uniformly cts. (Remark. Since $\varepsilon > 0$ is arbitrary, $\varepsilon/2$ in (1), (2) and (3) can be replaced by ε.

Remark. If, additionally, f is assumed to be cts on $\mathbb{R}$ then one can prove as follows:

Let $\varepsilon > 0$. Take $\begin{cases} a < -1 \\ b > 1 \end{cases}$ ~~such that~~ (or simply $b > a$)

such that

① $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1, x_2 \in (-\infty, a]$ ~~and~~

and ② $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1, x_2 \in [b, \infty)$.

Thanks to the continuity assumption, one can apply the Unif Cts Th to the interval $[a-1, b+1]$ to find $\delta > 0$ s.t.

③ $|f(x_1) - f(x_2)| < \varepsilon$ whenever $x_1, x_2 \in [a-1, b+1]$ with $|x_1 - x_2| < \delta$

Let $\delta' = \delta \wedge 1$. Let $u, x \in \mathbb{R}$ with $|u - x| < \delta'$.

It suffices to show that

(*) $|f(u) - f(x)| < \varepsilon$

This is done for each of the following cases:

(i) At least one of u, x lies in $\mathbb{R} \setminus [a-1, b+1]$ — Note then that both u, x should either be in $(-\infty, a]$ or be in $[b, \infty)$

so $(*)$ holds in such case by ① & ②.

(ii) The remaining case is: none of u, x lies in $\mathbb{R} \setminus [a-1, b+1]$, that is both u, x belongs $[a-1, b+1]$ — then $(*)$ holds by ③

Let $f: \mathbb{R} \to \mathbb{R}$ be cts and of period $p > 0$: $f(x+p) = x$ $\forall x \in \mathbb{R}$. Then f is unif. cts.

Proof. (1) Show first that $\forall x \in \mathbb{R}$, $\exists\, x' \in [0, p)$ s.t. $x \sim x'$ in the sense that $x - x' = np$ for some $n \in \mathbb{Z}$ (so "$\sim$" is an "equivalence relation"). To do this, we may assume that $x < 0$ (the case that $x \geq p$ can be dealt with similarly while the remaining case, $x \in [0, p)$, being trivial with $x' = x$ then).

By the Archimedean property and the well-order principle, take smallest $m \in \mathbb{N}$ s.t. $x + mp \geq 0$ (so $x + (m-1)p < 0$ and $x + mp < p$); then set $x' = x + mp$.

(2) Let $\varepsilon > 0$. By the Unif. Continuity theo (applied to $[-p, 2p]$), $\exists\, \delta > 0$ s.t.

① $|f(x') - f(u')| < \varepsilon$ whenever $|x' - u'| < \delta$ & $x', u' \in [-p, 2p]$.

Claim that

② $|f(x) - f(u)| < \varepsilon$ whenever $|x - u| < \underbrace{\delta \wedge p}_{\min\{\delta, p\}}$ & $x, u \in \mathbb{R}$.

To verify this, let $x, u \in \mathbb{R}$ with $|x - u| < \delta \wedge p$. By part (1), $\exists\, x' \in [0, p)$ s.t. $x \sim x'$: $x' = x - np$ for some $n \in \mathbb{Z}$. Let $u' \stackrel{\text{def}}{=} u - np$. Then $|x' - u'| = |x - u| < \delta \wedge p$ (so $|f(x') - f(u')| = |f(x) - f(u)|$ thanks to the periodicity); moreover, noting $x' \in [0, p)$ and $|x' - u'| < \delta \wedge p \leq p$, so $u' \in [-p, 2p)$ one has from ① that $|f(x') - f(u')| < \varepsilon$ i.e. $|f(x) - f(u)| < \varepsilon$.