Math 1030 Chapter 8

Reference.

Beezer, Ver 3.5 Section HSE (print version p44 - p50)Section NM (print version p51 - p56)

Exercise.

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf

1. Section HSE (ex p.18-23) C21-C23, C25-C27, C30-C31, M50-M52, T10-T12, T20
2. Section NM (ex p.23-27) C30-C33, C50, M30, M51-M52, T10, T12, T30, T31, T90.

8.1 Solutions of Homogeneous Systems

Definition 8.1 (Homogeneous System)**.** A system of linear equations, $\mathcal{LS}(A, \mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, if $\mathbf{b} = \mathbf{0}$, i.e.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Definition 8.2 (Homogeneous System corresponding to system of linear equation). The **homogeneous system** corresponding to $\mathcal{LS}(A, \mathbf{b})$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is $\mathcal{LS}(A, \mathbf{0})$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Example 8.3. The following is a homogeneous system of linear equations:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 0 \\ x_2 - x_4 &= 0 \\ x_1 + 3x_2 - 5x_3 + 5x_4 &= 0 \end{aligned}$$

It is the homogeneous system of linear equations corresponding to

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 1 \\ x_2 - x_4 &= 2 \\ x_1 + 3x_2 - 5x_3 + 5x_4 &= 3 \end{aligned}$$

Theorem 8.4 (Homogeneous Systems are Consistent)**.** *Suppose that a system of linear equations is homogeneous. Then the system is consistent. In fact* $\mathbf{0}$ *is a solution, i.e* $x_1 = x_2 = \cdots = x_n = 0$ *is a solution. Such solution is called a* **trivial solution** .

Proof. Set each variable of the system to zero. The left hand side of the all equations are zero, which are equal to the right hand side. □

Example 8.5. 1.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= 0 \\ 5x_1 + 5x_2 + 7x_3 &= 0 \\ x_1 + 4x_3 &= 0 \end{aligned}$$

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

It has $n - r = 3 - 3 = 0$ free variable. Hence it has only one solution.

2.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$$

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent. It has $n - r = 3 - 2 = 1$ free variable. The solution set is

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \; x_1 = -x_3, \; x_2 = x_3 \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \middle| \; x_3 \text{ real number} \right\}$$

Geometrically, these are points in three dimensions that lie on a line through the origin.

3.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

The reduced row echelon form of the augmented matrix is

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent. It has $n - r = 4 - 2 = 2$ free variables. The solution set is

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_1 = -3x_3 + 2x_4, \; x_2 = -x_3 + 3x_4 \right\}$$

$$= \left\{ \begin{bmatrix} -3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_3, \; x_4 \text{ real numbers} \right\}$$

Notice that when we do row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any one of the three allowable row operations will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So in this case, we may be as likely to reference only the coefficient matrix and presume that we remember that the final column begins with zeros, and after any number of row operations is still zero.

Theorem 8.6. *Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.*

Proof. The system is homogeneous, by theorem Homogeneous Systems are Consistent it is consistent. Then the hypothesis that $n > m$, together with Consistent, More Variables than Equations, Infinite solutions, gives infinitely many solutions. □

If $n = m$, then we can have a unique solution or infinitely many solutions (see the above examples).

8.2 Null Space of a Matrix

Definition 8.7. The **null space** of a an $m \times n$ matrix A, denoted by $\mathcal{N}(A)$, is the set of vectors $\vec{x} \in \mathbb{R}^n$ such that:

$$A\vec{x} = \vec{0}.$$

Equivalently, it is the set of all the vectors which are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. That is, if:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

then $\mathcal{N}(A)$ is the solution set of

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

Example 8.8. Suppose

$$A = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

Then

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -4 \\ 1 \\ -3 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

are in $\mathcal{N}(A)$ as $A\mathbf{x} = \mathbf{0}$, $A\mathbf{y} = \mathbf{0}$.

However, the vector

$$\mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

is not in $\mathcal{N}(A)$ as

$$A\mathbf{z} = \begin{bmatrix} -17 \\ 16 \\ -16 \\ 73 \end{bmatrix} \neq \mathbf{0}.$$

Example 8.9. Let us compute the null space of

$$A = \begin{bmatrix} 2 & -1 & 7 & -3 & -8 \\ 1 & 0 & 2 & 4 & 9 \\ 2 & 2 & -2 & -1 & 8 \end{bmatrix}$$

which we write as $\mathcal{N}(A)$. Translating Definition 8.7, we simply desire to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. So we row-reduce the augmented matrix to obtain:

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & \boxed{1} & -3 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \end{array}\right]$$

The variables (of the homogeneous system) x_3 and x_5 are free (since columns 1, 2 and 4 are pivot columns), so we arrange the equations represented by the matrix in reduced row-echelon form to:

$$\begin{aligned} x_1 &= -2x_3 - x_5 \\ x_2 &= 3x_3 - 4x_5 \\ x_4 &= -2x_5 \end{aligned}$$

So we can write the infinite solution set as sets using column vectors,

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} -2x_3 - x_5 \\ 3x_3 - 4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} \middle| \; x_3,\, x_5 \text{ real numbers} \right\}.$$

Example 8.10. Let us compute the null space of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

which we write as $\mathcal{N}(C)$. Translating definition Definition 8.7, we simply desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So we row-reduce the augmented matrix to obtain

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

There are no free variables in the homogeneous system represented by the row-reduced matrix, so there is only the trivial solution, the zero vector, $\mathbf{0}$. So we can write the (trivial) solution set as

$$\mathcal{N}(C) = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

8.3 Augmented matrix vs Coefficient Matrix

The augmented matrix for the homogeneous of system of linear equations $\mathcal{LS}(A, \mathbf{0})$ is $[A|\mathbf{0}]$. Any row operators on $[A|\mathbf{0}]$ will not change the last zero columns. If

$$A \xrightarrow{\text{row operations}} B$$

then

$$[A|\mathbf{0}] \xrightarrow{\text{same row operations}} [B|\mathbf{0}].$$

Therefore, for the homogeneous system of linear equations, we can replace the augmented matrix by the coefficient matrix. Just remember there is actually a zero column as the last column. For example:

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

We can start with coefficient matrix:

$$A = \begin{bmatrix} 2 & 1 & 6 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix}$$

The RREF is:

$$\begin{bmatrix} \boxed{1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding augmented matrix is:

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent. It has $n - r = 4 - 2 = 2$ free variables. The solution set is:

$$\begin{aligned} S &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_1 = -3x_3 + 2x_4,\; x_2 = -x_3 + 3x_4 \right\} \\ &= \left\{ \begin{bmatrix} -3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \; x_3,\; x_4 \text{ real numbers} \right\} \end{aligned}$$

8.4 Nonsingular Matrices

In this section we specialize further and consider matrices with equal numbers of rows and columns, which when considered as coefficient matrices lead to systems with equal numbers of equations and variables.

Definition 8.11 (Square Matrix)**.** A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular** .

Definition 8.12 (Nonsingular Matrix)**.** Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, in other words, the system has only the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix.

Example 8.13. 1. Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The system of linear equations $\mathcal{LS}(A, \mathbf{0})$ has nontrivial solutions. Hence A is singular.

2. Let
$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$
The system of linear equations $\mathcal{LS}(A, \mathbf{0})$ has only trivial solutions. So it is nonsingular.

Recall:

Definition 8.14 (Identity Matrix)**.** The $m \times m$ **identity matrix** , I_m, is defined by
$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i,\, j \leq m$$
i.e.
$$I_m = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Example 8.15. The 4×4 identity matrix is:
$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that an identity matrix is square, and in reduced row-echelon form. Also, every column is a pivot column, and every possible pivot column appears once.

Theorem 8.16 (Nonsingular Matrices Row Reduce to the Identity Matrix)**.** *Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.*

Proof. ($\Leftarrow$) Suppose B is the identity matrix. When the augmented matrix $[A|\mathbf{0}]$ is row-reduced, the result is $[B|\mathbf{0}] = [I_n|\mathbf{0}]$. The number of nonzero rows is equal to the number of variables in the linear system of equations $\mathcal{LS}(A, \mathbf{0})$, so $n = r$ and has $n - r = 0$ free variables. Thus, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has just one solution, which must be the trivial solution. This is exactly the definition of a nonsingular matrix.

($\Rightarrow$) If A is nonsingular, then the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution, and has no free variables in the description of the solution set. The homogeneous system is consistent, by Lecture 4 Theorem 4, the homogeneous system has $n - r$ free variables. Thus, $n - r = 0$, and so $n = r$. So B has n pivot columns among its total of n columns. This is enough to force B to be the $n \times n$ identity matrix I_n (why?). $\square$

Example 8.17.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is row equivalent to the reduced row echelon form

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since B is not the 3×3 identity matrix, the above theorem tells us that A is a singular matrix.

Example 8.18.

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

It is row-equivalent to the reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since B is the 3×3 identity matrix, A is a nonsingular matrix by the above theorem.

8.5 Null Space of a Nonsingular Matrix

Theorem 8.19 (Nonsingular Matrices have Trivial Null Spaces)**.** *Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A is the set containing only the zero vector, i.e., $\mathcal{N}(A) = \{\mathbf{0}\}$.*

Proof. The null space of a square matrix, A, is equal to the set of solutions to the homogeneous system, $\mathcal{LS}(A, \mathbf{0})$. A matrix is nonsingular if and only if the set of solutions to the homogeneous system, $\mathcal{LS}(A, \mathbf{0})$, has only a trivial solution. These two observations may be chained together to construct the two proofs necessary for each half of this theorem. □

Remark. It is now easy to see that any nonsingular matrix A is invertible:

If an $n \times n$ matrix A is invertible, then it is row-equivalent to I_n, which means there exists a sequence of elementary matrices $J_1, J_2, \ldots, J_l$ (each correspond to a row operation), such that:

$$J_l \cdots J_2 J_1 A = I.$$

Let $J = J_l \cdots J_2 J_1$. Since each elementary matrix is invertible, and the product of invertible matrices is also invertible, the matrix J is invertible.

Hence,

$$A = J^{-1} J A = J^{-1} I = J^{-1}$$

The matrix J^{-1}, being the inverse of an invertiable matrix, is invertible. We conclude that $A = J^{-1}$ is invertible.

(Observe that every nonsingular matrix is in particular a product of elementary matrices.)

Theorem 8.20 (Nonsingular Matrices and Unique Solutions)**.** *Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector* $\mathbf{b}$.

Proof. ($\Rightarrow$) The hypothesis for this half of the proof is that the system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$. We will make a very specific choice for $\mathbf{b}$: $\mathbf{b} = \mathbf{0}$. Then we know that the system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. But this is precisely the definition of what it means for A to be nonsingular.

($\Leftarrow$) We assume that A is nonsingular of size $n \times n$, so we know there is a sequence of row operations that will convert A into the identity matrix I_n (Theorem Nonsingular Matrices Row Reduce to the Identity Matrix). Form the augmented matrix $A' = [A|\mathbf{b}]$ and apply this same sequence of row operations to A'. The result will be the matrix $B' = [I_n|\mathbf{c}]$, which is in reduced row-echelon form with $r = n$. Then the augmented matrix B' represents the (extremely simple) system of equations $x_i = [\mathbf{c}]_i$, $1 \leq i \leq n$. The vector $\mathbf{c}$ is clearly a solution, so the system is consistent. With a consistent system, we use Lecture 4 Theorem 4 to count free variables. We find that there are $n - r = n - n = 0$ free variables, and so we therefore know that the solution is unique. □

Alternatively,

Proof. If A is nonsingular, then A^{-1} exists. So, $A\mathbf{x} = \mathbf{b}$ implies that the unique solution is $\mathbf{x} = A^{-1}\mathbf{b}$.

Conversely, if $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b}$. Then, in particular, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. This implies by definition that A is nonsingular. □

Theorem 8.21 (Nonsingular Matrix Equivalences)**.** *Suppose that A is a square matrix. The following are equivalent.*

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

Proof. The statement that A is nonsingular is equivalent to each of the subsequent statements by, in turn, theorems Nonsingular Matrices Row Reduce to the Identity Matrix, Nonsingular Matrices have Trivial Null Spaces, Nonsingular Matrices and Unique Solutions. So the statement of this theorem is just a convenient way to organize all these results. □

In fact, we further have: Nonsingular Matrix Equivalences, Round 3.

8.6 Particular Solutions, Homogeneous Solutions

The next theorem tells us that in order to find all of the solutions to a linear system of equations, it is sufficient to find just one solution, and then find all of the solutions to the corresponding homogeneous system. This explains part of our interest in the null space, the set of all solutions to a homogeneous system.

Theorem 8.22 (Particular Solution Plus Homogeneous Solutions)**.** *Suppose that $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$. Then $\mathbf{y}$ is a solution to $\mathcal{LS}(A, \mathbf{b})$ if and only if $\mathbf{y} = \mathbf{w} + \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$, i.e.*

1. *If $\mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $\mathbf{y} - \mathbf{w} \in \mathcal{N}(A)$*
2. *If $\mathbf{z} \in \mathcal{N}(A)$, then $\mathbf{w} + \mathbf{z}$ is a solution of $A\mathbf{x} = \mathbf{b}$*

In other words, there is a one-to-one correspondence between

$$\textit{solution set of } A\mathbf{x} = \mathbf{b} \longleftrightarrow \mathcal{N}(A)\,,$$

through

$$\mathbf{y} \to \mathbf{y} - \mathbf{w},$$

$$\mathbf{w} + \mathbf{z} \leftarrow \mathbf{z}.$$

Proof. Because $\mathbf{w}$ is one solution to the linear system of equations $\mathcal{LS}(A, \mathbf{b})$, $A\mathbf{w} = \mathbf{b}$.

1. If $\mathbf{y}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{y} = \mathbf{b}$. Hence $A(\mathbf{y}-\mathbf{w}) = A\mathbf{y} - A\mathbf{w} = \mathbf{b} - \mathbf{b} = \mathbf{0}$. So $\mathbf{y} - \mathbf{w} \in \mathcal{N}(A)$.

2. Suppose $\mathbf{z} \in \mathcal{N}(A)$, $A\mathbf{z} = \mathbf{0}$. So $A(\mathbf{w} + \mathbf{z}) = A\mathbf{w} + A\mathbf{z} = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Hence $\mathbf{w} + \mathbf{z}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

□

Example 8.23.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

is a consistent system of equations with a nontrivial null space. Let A denote the coefficient matrix of this system.

Consider the following three solutions to the system:

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{y}_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{y}_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}$$

Let $\mathbf{w} = \mathbf{y}_1$. Then,

$$\mathbf{y}_2 - \mathbf{w} = \begin{bmatrix} 4 \\ -1 \\ -2 \\ -1 \end{bmatrix}, \quad \mathbf{y}_3 - \mathbf{w} = \begin{bmatrix} 7 \\ 7 \\ -1 \\ 2 \end{bmatrix}$$

are indeed elements in $\mathcal{N}(A)$ (check!).

To find all the solutions, we may first work out (using Gaussian elimination on $[A|\mathbf{0}]$, for example) that:

$$\mathcal{N}(A) = \left\{ x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_3,\, x_4 \in \mathbb{R} \right\}$$

By the theorem, the solution set to the linear system is:

$$\mathbf{w} + \mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_3,\, x_4 \in \mathbb{R} \right\}$$