Math 1030 Chapter 17

Reference.

Beezer, Ver 3.5 Section MISLE, Section MINM (print version p149 - p161)

Exercise.

Exercises with solutions can be downloaded at http://linear.ups.edu/download/fcla-3.50-solution-manual.pdfSection MISLE (p60-64), all. Section MINM C20, C40, M10, M11, M15, M80, T25.

17.1 Solution Inverse

The inverse of a square matrix, and solutions to linear systems with square coefficient matrices, are intimately connected.

Example 17.1.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

We can represent this system of equations as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$$

Now, entirely unmotivated, we define the 3×3 matrix B,

$$\begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

and note the remarkable fact that

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now apply this computation to the problem of solving the system of equations,

$$\mathbf{x} = I_3\mathbf{x} = (BA)\mathbf{x} = B(A\mathbf{x}) = B\mathbf{b}$$

So we have

$$\mathbf{x} = B\mathbf{b} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

So with the help and assistance of B we have been able to determine a solution to the system represented by $A\mathbf{x} = \mathbf{b}$ through judicious use of matrix multiplication. Since the coefficient matrix in this example is nonsingular, there would be a unique solution, no matter what the choice of $\mathbf{b}$. The derivation above amplifies this result, since we were forced to conclude that $\mathbf{x} = B\mathbf{b}$ and the solution could not be anything else. You should notice that this argument would hold for any particular choice of $\mathbf{b}$.

The matrix B of the previous example is called the inverse of A. When A and B are combined via matrix multiplication, the result is the identity matrix, which can be inserted *in front* of $\mathbf{x}$ as the first step in finding the solution. This is entirely analogous to how we might solve a single linear equation like $3x = 12$.

$$x = 1x = \left(\frac{1}{3}(3)\right)x = \frac{1}{3}(3x) = \frac{1}{3}(12) = 4$$

Here we have obtained a solution by employing the **multiplicative inverse** of 3, $3^{-1} = \frac{1}{3}$. This works fine for any scalar multiple of x, except for zero, since zero does not have a multiplicative inverse. Consider separately the two linear equations,

$$0x = 12 \qquad\qquad 0x = 0$$

The first has no solutions, while the second has infinitely many solutions. For matrices, it is all just a little more complicated. Some matrices have inverses, some do not. And when a matrix does have an inverse, just how would we compute it? In other words, just where did that matrix B in the last example come from? Are there other matrices that might have worked just as well?

17.2 Inverse of a Matrix

Definition 17.2. Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.

Notice that if B is the inverse of A, then we can just as easily say A is the inverse of B, or A and B are inverses of each other.

Not every square matrix has an inverse.

Example 17.3. A matrix without an inverse Consider the coefficient matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Suppose that A is invertible and does have an inverse, say B. Choose the vector of constants

$$\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and consider the system of equations $A\mathbf{x} = \mathbf{b}$. Just as in the previous example, this vector equation would have the unique solution $\mathbf{x} = B\mathbf{b}$.

However, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Form the augmented matrix $[A|\mathbf{b}]$ and row-reduce to

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

which allows us to recognize the inconsistency.

So the assumption of A's inverse leads to a logical inconsistency (the system cannot be both consistent and inconsistent), so our assumption is false. A is not invertible.

Let us look at one more matrix inverse before we embark on a more systematic study.

Example 17.4. Matrix inverse1.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix},$$

Then

$$AB = BA = I_2.$$

So B is the inverse of A.

2.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

Then

$$AB = BA = I_3.$$

So B is the inverse of A.

3. Consider the matrices,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so by the definition of inverse matrix, we can say that A is invertible and write $B = A^{-1}$.

We will now concern ourselves less with whether or not an inverse of a matrix exists, but instead with how you can find one when it does exist. Later we will have some theorems that allow us to more quickly and easily determine just when a matrix is invertible.

Theorem 17.5 (Two-by-Two Matrix Inverse)**.** *Suppose*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof. $\Leftarrow$ Assume that $ad - bc \neq 0$. We will use the definition of the inverse of a matrix to establish that A has an inverse. Note that if $ad - bc \neq 0$ then the displayed formula for A^{-1} is legitimate since we are not dividing by zero). Using this proposed formula for the inverse of A, we compute

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right) = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This is sufficient to establish that A is invertible, and that the expression for A^{-1} is correct.

$\Rightarrow$ Assume that A is invertible, and proceed with a proof by contradiction, by assuming also that $ad - bc = 0$. This translates to $ad = bc$. Let

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

be a putative inverse of A.

This means that

$$I_2 = AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Working on the matrices on two ends of this equation, we will multiply the top row by c and the bottom row by a.

$$\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ace+bcg & acf+bch \\ ace+adg & acf+adh \end{bmatrix}$$

We are assuming that $ad = bc$, so we can replace two occurrences of ad by bc in the bottom row of the right matrix.

$$\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ace+bcg & acf+bch \\ ace+bcg & acf+bch \end{bmatrix}$$

The matrix on the right now has two rows that are identical, and therefore the same must be true of the matrix on the left. Identical rows for the matrix on the left implies that $a = 0$ and $c = 0$.

With this information, the product AB becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} bg & bh \\ dg & dh \end{bmatrix}$$

So $bg = dh = 1$ and thus b, g, d, h are all nonzero. But then bh and dg (the **other corners**) must also be nonzero, so this is (finally) a contradiction. So our assumption was false and we see that $ad - bc \neq 0$ whenever A has an inverse. □

There are several ways one could try to prove this theorem, but there is a continual temptation to divide by one of the eight entries involved (a through f), but we can never be sure if these numbers are zero or not. This could lead to an analysis by cases, which is messy, messy, messy. Note how the above proof never divides, but always multiplies, and how zero/nonzero considerations are handled. Pay attention to the expression $ad - bc$, as we will see it again in a while.

This theorem is cute, and it is nice to have a formula for the inverse, and a condition that tells us when we can use it. However, this approach becomes impractical for larger matrices, even though it is possible to demonstrate that, in theory, there is a general formula. (Think for a minute about extending this result to just 3×3 matrices. For starters, we need 18 letters!) Instead, we will work column-by-column. Let us first work an example that will motivate the main theorem and remove some of the previous mystery.

17.3 Nonsingular Matrices are Invertible

For $\alpha, \beta \in \mathbb{R}$, then $\alpha\beta \neq 0$ if and only if $\alpha \neq 0$ and $\beta \neq 0$. We have a similar result for nonsingular matrix

Theorem 17.6 (Nonsingular Product has Nonsingular Terms)**.** *Suppose that A and B are square matrices of size n. The product AB is nonsingular if and only if A and B are both nonsingular.*

Proof. ($\Rightarrow$) For this portion of the proof we will form the logically-equivalent contrapositive and prove that statement using two cases.

AB is nonsingular implies A and B are both nonsingular. becomes

A or B is singular implies AB is singular.

Case 1. Suppose B is singular. Then there is a nonzero vector $\mathbf{z}$ that is a solution to $B\mathbf{x} = \mathbf{0}$. So

$$\begin{aligned}(AB)\mathbf{z} &= A(B\mathbf{z}) \\ &= A\mathbf{0} \\ &= \mathbf{0}\end{aligned}$$

Then $\mathbf{z}$ is a nonzero solution to $AB\mathbf{x} = \mathbf{0}$. Thus AB is singular as desired.

Case 2. Suppose A is singular, and B is not singular. Because A is singular, there is a nonzero vector $\mathbf{y}$ that is a solution to $A\mathbf{x} = \mathbf{0}$. Now consider the linear system $B\mathbf{x} = \mathbf{y}$. Since B is nonsingular, the system has a unique solution, which we will denote as $\mathbf{w}$. We first claim $\mathbf{w}$ is not the zero vector either. Assuming the opposite, suppose that $\mathbf{w} = \mathbf{0}$. Then

$$\begin{aligned}\mathbf{y} &= B\mathbf{w} \\ &= B\mathbf{0} \\ &= \mathbf{0}\end{aligned}$$

$$\begin{aligned}(AB)\mathbf{w} &= A(B\mathbf{w}) \\ &= A\mathbf{y} \\ &= \mathbf{0}\end{aligned}$$

So $\mathbf{w}$ is a nonzero solution to $AB\mathbf{x} = \mathbf{0}$. Thus AB is singular as desired. And this conclusion holds for both cases.

($\Leftarrow$) Now assume that both A and B are nonsingular. Suppose that $\mathbf{x} \in \mathbb{R}^n$ is a solution to $AB\mathbf{x} = \mathbf{0}$. Then

$$\begin{aligned}\mathbf{0} &= (AB)\,\mathbf{x} \\ &= A\,(B\mathbf{x})\end{aligned}$$

So $B\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{0}$, and by the definition of a nonsingular matrix, we conclude that $B\mathbf{x} = \mathbf{0}$. Now, by an entirely similar argument, the nonsingularity of B forces us to conclude that $\mathbf{x} = \mathbf{0}$. So the only solution to $AB\mathbf{x} = \mathbf{0}$ is the zero vector and we conclude that AB is nonsingular. $\square$

The contrapositive of this entire result is equally interesting. It says that A or B (or both) is a singular matrix if and only if the product AB is singular.

Theorem 17.7 (One-Sided Inverse is Sufficient)**.** *Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.*

Proof. The matrix I_n is nonsingular. So A and B are nonsingular by, so in particular A is nonsingular.

Hence, A is row equivalent to the identity matrix I_n, which means there are elementary matrices $J_1, J_2, \ldots, J_k$ such that:

$$J_k \cdots J_2 J_1 A = I_n.$$

Hence, $AB = I_n$ implies that:

$$\underbrace{J_k \cdots J_2 J_1 A}_{I_n} B = \underbrace{J_k \cdots J_2 J_1 I_n}_{J_k \cdots J_2 J_1}$$

So, $B = J_k \cdots J_2 J_1$, and:

$$BA = J_k \cdots J_2 J_1 A = I_n$$

□

Theorem 17.8 (Nonsingularity is Invertibility)**.** *Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.*

Proof. ($\Leftarrow$) Suppose A is invertible. Then, $A\mathbf{x} = \mathbf{0}$ if and only if :

$$\underbrace{A^{-1}A\mathbf{x}}_{\mathbf{x}} = A^{-1}\mathbf{0} = \mathbf{0}.$$

This implies that the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$ only exactly one unique solution $\mathbf{x} = \mathbf{0}$. Hence, A is nonsingular.

($\Rightarrow$) Suppose A is a nonsingular $n \times n$ matrix. Then, A is row-equivalent to I_n, which means there are elementary matrices $J_1, J_2, \ldots, J_k$ such that:

$$J_k \cdots J_2 J_1 A = I_n$$

It now follows from One-Sided Inverse is Sufficient that A is invertible, with:

$$A^{-1} = J_k \cdots J_2 J_1.$$

□

Remark. So for a square matrix, the properties of having an inverse and of having a trivial null space are one and the same. Cannot have one without the other.

17.4 Computing the Inverse of a Matrix

Theorem 17.9 (Computing the Inverse of a Nonsingular Matrix)**.** *Suppose A is a nonsingular square matrix of size n. Create the $n \times 2n$ matrix M by placing the $n \times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $JA = AJ = I_n$. Hence, $A^{-1} = J$.*

Remark. Observe this procedure also allows one to see whether a given matrix A in nonsingular.

Proof. Since A is nonsingular, there exist a sequence of row operations $R_1, R_2, \ldots R_k$ such that:

$$A \xrightarrow{R_1} \cdots \xrightarrow{R_2} \cdots \xrightarrow{R_k} I_n$$

Recall that each row operation R_i corresponds to multiplcation by an elementary matrix J_i from the left, that is:

$$A \xrightarrow{R_1} J_1 A \xrightarrow{R_2} J_2 J_1 A \xrightarrow{R_3} \cdots \xrightarrow{R_k} J_k \cdots J_2 J_1 A = I_n.$$

Start with the augmented matrix:

$$[A|I_n]$$

Applying the row operation R_1 to the matrix above is equivalent to:

$$J_1[A|I_n] = [J_1 A | J_1 I_n] = [J_1 A | J_1]$$

Further applying the row operation R_2 gives:

$$J_2[J_1 A | J_1 I_n] = [J_2 J_1 A | J_2 J_1]$$

Applying the rest of the row operations which reduce A to I_n, we have:

$$[\underbrace{J_k \cdots J_2 J_1 A}_{I_n} \,|\, \underbrace{J_k \cdots J_2 J_1}_{J = A^{-1}}]$$

□

Example 17.10 (Computing a matrix inverse)**.** Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find A^{-1}.

$$[A|\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-1R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{1R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{-1R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{1R_3+R_2, -1R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array}\right]$$

So

$$B = [\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3] = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 1 & -1 & -1 \end{bmatrix}$$

Example 17.11 (Computing a matrix inverse)**.** Let

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}.$$

Find B^{-1}.

$$M = \left[\begin{array}{ccc|ccc} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right].$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & -12 & -9 \\ 0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2} \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}.$$

Theorem 17.12 (Nonsingular Matrix Equivalences Round 3)**.** *Suppose that A is a square matrix of size n. The following are equivalent.*

1. *A is nonsingular.*

2. *A row-reduces to the identity matrix.*

3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*

4. *The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every possible choice of $\mathbf{b}$.*

5. *The columns of A are a linearly independent set.*

6. *A is invertible.*

In the case that A is a nonsingular coefficient matrix of a system of equations, the inverse allows us to very quickly compute the unique solution, for any vector of constants.

Theorem 17.13 (Solution with Nonsingular Coefficient Matrix)**.** *Suppose that A is nonsingular. Then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $A^{-1}\mathbf{b}$.*

Proof. We can show this by simply plug $A^{-1}\mathbf{b}$ in the solution.

$$\begin{aligned} A\left(A^{-1}\mathbf{b}\right) &= \left(AA^{-1}\right)\mathbf{b} \\ &= I_n\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

Since $A\mathbf{x} = \mathbf{b}$ is true when we substitute $A^{-1}\mathbf{b}$ for $\mathbf{x}$, $A^{-1}\mathbf{b}$ is a (the!) solution to $A\mathbf{x} = \mathbf{b}$. □

Example 17.14. Using the previous theorem, solve

$$\begin{aligned} x_1 + x_2 - x_3 + 4x_4 &= 1 \\ x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\ 2x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + 2x_3 - 9x_4 &= -1 \end{aligned}$$

The matrix coefficient is

$$A = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & -9 \end{bmatrix}.$$

After some computations,

$$A^{-1} = \begin{bmatrix} -33 & -22 & 45 & -17 \\ 35 & 23 & -47 & 18 \\ 25 & 17 & -34 & 13 \\ 6 & 4 & -8 & 3 \end{bmatrix}.$$

Then the solution of the system of linear equations is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -60 \\ 63 \\ 46 \\ 11 \end{bmatrix}.$$

17.5 Properties of Matrix Inverses

The inverse of a matrix enjoys some nice properties. We collect a few here. First, a matrix can have but one inverse.

Theorem 17.15 (Matrix Inverse is Unique)**.** *Suppose the square matrix A has an inverse. Then A^{-1} is unique.*

Proof. We will assume that A has two inverses. The hypothesis tells there is at least one. Suppose then that B and C are both inverses for A. Then $AB = BA = I_n$ and $AC = CA = I_n$. Then we have,

$$\begin{aligned} B &= BI_n \\ &= B(AC) \\ &= (BA)C \\ &= I_nC \\ &= C \end{aligned}$$

So we conclude that B and C are the same, and cannot be different. So any matrix that acts like an inverse, must be the inverse. □

When most of us dress in the morning, we put on our socks first, followed by our shoes. In the evening we must then first remove our shoes, followed by our socks. Try to connect the conclusion of the following theorem with this everyday example.

Theorem 17.16 (Socks and Shoes)**.** *Suppose A and B are invertible matrices of size n. Then AB is an invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$.*

Proof.

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}I_nB \\ &= B^{-1}B \\ &= I_n\end{aligned}$$

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} \\ &= I_n\end{aligned}$$

So the matrix $B^{-1}A^{-1}$ has met all of the requirements to be AB's inverse (date) and with the ensuing marriage proposal we can announce that $(AB)^{-1} = B^{-1}A^{-1}$. □

Theorem 17.17 (Matrix Inverse of a Matrix Inverse)**.** *Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.*

Proof. As with the proof of of the previous example, we examine if A is a suitable inverse for A^{-1} (by definition, the opposite is true).

$$AA^{-1} = I_n$$

$$A^{-1}A = I_n$$

The matrix A has met all the requirements to be the inverse of A^{-1}, and so is invertible and we can write $A = (A^{-1})^{-1}$. □

Theorem 17.18 (Matrix Inverse of a Transpose)**.** *Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.*

Proof. As with the proof of Theorem Socks and Shoes, we see if $(A^{-1})^t$ is a suitable inverse for A^t.

$$\begin{aligned}(A^{-1})^tA^t &= (AA^{-1})^t \\ &= I_n^t \\ &= I_n\end{aligned}$$

$$\begin{aligned}A^t(A^{-1})^t &= (A^{-1}A)^t \\ &= I_n^t \\ &= I_n\end{aligned}$$

The matrix $(A^{-1})^t$ has met all the requirements to be the inverse of A^t, and so is invertible and we can write $(A^t)^{-1} = (A^{-1})^t$. □

Theorem 17.19 (Matrix Inverse of a Scalar Multiple)**.** *Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ and αA is invertible.*

Proof. As with the proof of Theorem Socks and Shoes, we see if $\frac{1}{\alpha} A^{-1}$ is a suitable inverse for αA.

$$\begin{aligned}\left(\frac{1}{\alpha} A^{-1}\right)(\alpha A) &= \left(\frac{1}{\alpha}\alpha\right)\left(A^{-1}A\right) \\ &= 1 I_n \\ &= I_n\end{aligned}$$

$$\begin{aligned}(\alpha A)\left(\frac{1}{\alpha} A^{-1}\right) &= \left(\alpha\frac{1}{\alpha}\right)\left(AA^{-1}\right) \\ &= 1 I_n \\ &= I_n\end{aligned}$$

The matrix $\frac{1}{\alpha} A^{-1}$ has met all the requirements to be the inverse of αA, so we can write $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$. □

Notice that there are some likely theorems that are missing here. For example, it would be tempting to think that $(A + B)^{-1} = A^{-1} + B^{-1}$, but this is false. Can you find a counterexample?