

Math 1030 Chapter 12

Reference.

- Beezer, Ver 3.5 Section LI (print version p95 - p104)
- Strang, Section 2.3

Exercise.

- Exercises with solutions can be downloaded at <http://linear.ups.edu/download/fcla-3.50-solution-manual.pdf> Section LI (p.40-48) (Replace \mathbb{C} by \mathbb{R} in the following questions) C20-25, C30-33, C60, M20, M21, M50, M51, T10-13, T15, T20, T50.
- Strang, Section 2.3

Linear independence is one of the most fundamental conceptual ideas in linear algebra, along with the notion of **span**. So this lecture, and the subsequent one, will explore this new idea.

12.1 Linearly Independent Sets of Vectors

Definition 12.1 (Relation of Linear Dependence). Given a set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$, an equality of the form

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \dots + \alpha_n \mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S . If this equality is formed in a trivial fashion, i.e., $\alpha_i = 0, 1 \leq i \leq n$, then we say that it is the **trivial relation of linear dependence** on S .

Definition 12.2 (Linear Independence). The set of vectors $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the only relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors.

Remark. In short, a set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\}$ is linearly independent if and only if the only solution to:

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \dots + x_n\mathbf{u}_n = \mathbf{0}$$

is:

$$x_1 = x_2 = \dots = x_n = 0.$$

Theorem 12.3 (Linearly Independent Vectors and Homogeneous Systems). *Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$ is a set of vectors and that A is the $m \times n$ matrix whose columns are the vectors in S . Then S is a linearly independent set if and only if the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution.*

Proof. (\Leftarrow) Suppose that $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. Since it is a homogeneous system, this solution must be the trivial solution, $\mathbf{x} = \mathbf{0}$. This means that the only relation of linear dependence on S is the trivial one. So S is linearly independent.

(\Rightarrow) We will prove the contrapositive. Suppose that $\mathcal{LS}(A, \mathbf{0})$ does not have a unique solution. Since it is a homogeneous system, it is consistent. And so must have infinitely many solutions. One of these infinitely many solutions must be nontrivial (in fact, almost all of them are); choose one. This nontrivial solution will give a nontrivial relation of linear dependence on S . We therefore conclude that S is a linearly dependent set. \square

Since the above theorem is an "if-and-only-if" statement, we can use it to determine the linear independence or dependence of any set of column vectors, just by creating a matrix and analyzing its row-reduced echelon form. Let us illustrate this with two more examples.

Example 12.4. Linearly dependent set in \mathbb{R}^5

Consider the following set of $n = 4$ vectors in \mathbb{R}^5 :

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To determine linear independence, we first form an arbitrary relation of linear dependence,

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

We know that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is a solution to this equation, but that is of no interest whatsoever. That is always the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions.

In other words, are there nontrivial solutions to the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$, where the columns of A consist of the vectors in S .

Row-reducing the matrix A gives:

$$A = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We could solve the corresponding homogeneous system completely, but for this example all we need is one nontrivial solution. Setting the lone free variable to any nonzero value, such as $x_4 = 1$, yields the nontrivial solution:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}.$$

Hence,

$$2 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + (-4) \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}.$$

This is a relation of linear dependence on S that is not trivial, so we conclude that S is **linearly dependent**.

Example 12.5. Linearly independent set in \mathbb{R}^5

Consider the following set of $n = 4$ vectors in \mathbb{R}^5 :

$$T = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

To determine linear independence we first form an arbitrary relation of linear dependence,

$$\alpha_1 \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4 \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}.$$

We want to know if there are solutions to the equation above besides the trivial one: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$.

Row-reducing the associated matrix gives:

$$B = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the form of this matrix, we see that there are no free variables. Hence the associated homogeneous linear system has only the trivial solution. So we now know that there is but one way to combine the four vectors of T into a relation of linear dependence, and that this one way is the easy and obvious way. Hence, the set T is **linearly independent**.

12.1.1 More Examples

Example 12.6. Linearly independent

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ 5 \\ 1 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. The above theorem suggests that we study the matrix A whose columns are the vectors in S . Specifically, we are interested in the size of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Row-reducing A gives:

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $r = 3$, so there are $n - r = 3 - 3 = 0$ free variables. Hence $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. By the above theorem, the set S is linearly independent.

Example 12.7. Linearly dependent

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. Theorem Linearly Independent Vectors and Homogeneous Systems suggests that we study the matrix A whose columns are the vectors in S . Specifically, we are interested in the size of the solution set of the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Row-reducing A gives

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have $r = 2$, so there are $n - r = 3 - 2 = 1$ free variables. Hence $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. By Theorem Linearly Independent Vectors and Homogeneous Systems, the set S is linearly dependent.

Theorem Linearly Independent Vectors and Homogeneous Systems gives us a straightforward way to determine if a set of vectors is linearly independent or dependent.

Review the previous two examples. They are very similar, differing only in the last two slots of the third vector. This resulted in slightly different matrices when row-reduced, and different values of r , the number of nonzero rows. Notice, too, that we are less interested in the actual solution set, and more interested in its form or size. These observations allow us to make a slight improvement on Theorem Linearly Independent Vectors and Homogeneous Systems.

12.2 Relation between Linear Independence and the Number of Pivot Columns

Theorem 12.8 (Linearly Independent Vectors r and n). *Suppose that*

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots, \mathbf{v}_n\} \subseteq \mathbb{R}^m$$

is a set of vectors and that A is the $m \times n$ matrix whose columns are the vectors in S . Let B be a matrix in reduced row-echelon form that is row-equivalent to A and let r denote the number of pivot columns in B . Then S is linearly independent if and only if $n = r$.

Proof. Theorem Linearly Independent Vectors and Homogeneous Systems says the linear independence of S is equivalent to the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$ having a unique solution. Since the zero vector is a solution of $\mathcal{LS}(A, \mathbf{0})$, $\mathcal{LS}(A, \mathbf{0})$ is consistent. We can therefore apply Consistent Systems, r and n to see that the solution is unique exactly when $n = r$. \square

Here is an example of the most straightforward way to determine if a set of column vectors is linearly independent or linearly dependent. While this method can be quick and easy, do not forget the logical progression from the definition of linear independence through homogeneous system of equations which makes it possible.

Example 12.9. Linear dependence, r and n

Is the set of vectors:

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ -2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Solution. Theorem Linearly Independent Vectors, r and n suggests that we take the vectors of S as the columns of a matrix and then analyze its reduced row-echelon form:

$$\begin{bmatrix} 2 & 9 & 1 & -3 & 6 \\ -1 & -6 & 1 & 1 & -2 \\ 3 & -2 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 3 \\ 3 & 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we need only compute that $r = 4 < 5 = n$ to recognize, via Theorem Linearly Independent Vectors, r and n , that S is a linearly dependent set. Boom!

Example 12.10. Large linearly dependent set in \mathbb{R}^4

Consider the set of $n = 9$ vectors from \mathbb{R}^4 ,

$$R = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To employ Theorem Linearly Independent Vectors and Homogeneous Systems, we form a 4×9 matrix C whose columns are the vectors in R :

$$C = \begin{bmatrix} -1 & 7 & 1 & 0 & 5 & 2 & 3 & 1 & -6 \\ 3 & 1 & 2 & 4 & -2 & 1 & 0 & 1 & -1 \\ 1 & -3 & -1 & 2 & 4 & -6 & -3 & 5 & 1 \\ 2 & 6 & -2 & 9 & 3 & 4 & 1 & 3 & 1 \end{bmatrix}.$$

To determine if the homogeneous system $\mathcal{LS}(C, \mathbf{0})$ has a unique solution or not, we would normally row-reduce this matrix. But in this particular example, we can do better:

Since the system is homogeneous with $n = 9$ variables in $m = 4$ equations, and $n > m$, there are infinitely many solutions. Since there is not a unique solution, Theorem Linearly Independent Vectors and Homogeneous Systems says the set R is linearly dependent.

The following theorem generalizes the previous example.

Theorem 12.11 (More Vectors than Size implies Linear Dependence). *Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \dots, \mathbf{u}_n\} \subseteq \mathbb{R}^m$ and $n > m$. Then S is a linearly dependent set.*

Proof. Form the $m \times n$ matrix A whose columns are \mathbf{u}_i , $1 \leq i \leq n$. Consider the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. By Theorem 8.6 this system has infinitely many solutions. Since the system does not have a unique solution, Theorem Linearly Independent Vectors and Homogeneous Systems says the columns of A form a linearly dependent set, as desired. \square

12.3 Linear Independence and Nonsingular Matrices

We will now specialize to sets of n vectors in \mathbb{R}^n .

Example 12.12. Linearly dependent columns Do the columns of the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

form a linearly independent or dependent set?

Solution. We can show that A is singular. According to the definition of nonsingular matrices, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. So, by Theorem Linearly Independent Vectors and Homogeneous Systems, the columns of A form a linearly dependent set.

Example 12.13. Linearly independent columns

Do the columns of this matrix

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

form a linearly independent or dependent set?

Solution. We can show that B is nonsingular. According to the definition of nonsingular matrices, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. So, by Theorem Linearly Independent Vectors and Homogeneous Systems, the columns of B form a linearly independent set.

That the previous two examples have opposite properties for the columns of their coefficient matrices is no accident. Here is the theorem, and then we will update our equivalences for nonsingular matrices.

Theorem 12.14 (Nonsingular Matrices have Linearly Independent Columns). *Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.*

Proof. This is a proof where we can chain together equivalences, rather than proving the two halves separately.

$$\begin{aligned} A \text{ nonsingular} &\iff \mathcal{LS}(A, \mathbf{0}) \text{ has a unique solution} \\ &\iff A\vec{x} = \vec{0} \text{ has a unique solution } \vec{x} \\ &\iff \text{columns of } A \text{ are linearly independent} \end{aligned}$$

□

Here is the update to Nonsingular Matrix Equivalences

Theorem 12.15 (Nonsingular Matrix Equivalences Round 2). *Suppose that A is a square matrix. The following are equivalent.*

1. A is nonsingular.
2. A row-reduces to the identity matrix.
3. The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.
4. The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of \mathbf{b} .
5. The columns of A form a linearly independent set.

Proof. This follows directly from Nonsingular Matrices have Linearly Independent Columns and Nonsingular Matrix Equivalences, Round 2. \square

12.4 Null Spaces, Spans, Linear Independence

In this section, we will find a linearly independent set that spans a null space. Recall that, by Theorem 11.10, there exists a particular set of $n - r$ vectors that could be used to span the null space of a matrix.

Example 12.16. Linear independence of null space basis Suppose that we are interested in the null space of a 3×7 matrix A which row-reduces to

$$B = \begin{bmatrix} \boxed{1} & 0 & -2 & 4 & 0 & 3 & 9 \\ 0 & \boxed{1} & 5 & 6 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 8 & -5 \end{bmatrix}.$$

Then $F = \{3, 4, 6, 7\}$ is the set of indices for our four free variables that would be used in a description of the solution set for the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. Applying Theorem 8.6, we can begin to construct a set of four vectors whose span is the null space of A , a set of vectors we will refer to as T .

$$\begin{aligned} \mathcal{N}(A) = \langle T \rangle &= \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4\} \rangle \\ &= \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle \end{aligned}$$

So far, we have constructed as much of these individual vectors as we can, based just on the knowledge of the contents of the set F . This has allowed us to determine the entries in slots 3, 4, 6 and 7, while we have left slots 1, 2 and 5 blank. Without doing any more, let us ask if T is linearly independent? Begin with a relation of linear dependence on T , and see what we can learn about the scalars.

$$\begin{aligned}
\mathbf{0} &= \alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2 + \alpha_3 \mathbf{z}_3 + \alpha_4 \mathbf{z}_4 \\
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} &= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \alpha_2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \alpha_3 \\ 0 \\ \alpha_3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \alpha_4 \\ 0 \\ \alpha_4 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_3 \\ \alpha_4 \\ \alpha_4 \end{bmatrix}
\end{aligned}$$

Applying the equalities of vectors, we see that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. So the only relation of linear dependence on the set T is the trivial one. By the definition of linear independence, the set T is linearly independent. The important feature of this example is how the **pattern of zeros and ones** in the four vectors led to the conclusion of linear independence.

Theorem 12.17 (Basis for Null Spaces). *Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1, d_2, d_3, \dots, d_r\}$ and $F = \{f_1, f_2, f_3, \dots, f_{n-r}\}$ be the sets of column indices of B which are and are not, respectively, pivot columns. Construct the $n - r$ vectors \mathbf{z}_j , $1 \leq j \leq n - r$ of size n as*

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F, i = f_j \\ 0 & \text{if } i \in F, i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D, i = d_k \end{cases}$$

(In fact \mathbf{z}_j corresponding to the solution $x_{f_j} = 1$ and $x_{f_k} = 0$ for $k \neq j$.) Define the set $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$. Then

1. $\mathcal{N}(A) = \langle S \rangle$.

2. S is a linearly independent set.

Proof. Study the above example. You can skip the proof for now. Notice first that the vectors \mathbf{z}_j , $1 \leq j \leq n - r$, are the same as the $n - r$ vectors defined in Theorem 11.10. Also, the hypotheses of Theorem 11.10 are the same as the hypotheses of the theorem we are currently proving. So Theorem 11.10 tells us that $\mathcal{N}(A) = \langle S \rangle$. That was the easy half, but the second part is not much harder. What is new here is the claim that S is a linearly independent set.

To prove the linear independence of a set, we need to start with a relation of linear dependence and somehow conclude that the scalars involved must all be zero, i.e., that the relation of linear dependence is trivial. So, we start with an equation of the form

$$\alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2 + \alpha_3 \mathbf{z}_3 + \cdots + \alpha_{n-r} \mathbf{z}_{n-r} = \mathbf{0}.$$

For each j , $1 \leq j \leq n - r$, consider the equality of the individual entries of the vectors on both sides of this equality in position f_j :

$$\begin{aligned} 0 &= [\mathbf{0}]_{f_j} \\ &= [\alpha_1 \mathbf{z}_1 + \alpha_2 \mathbf{z}_2 + \alpha_3 \mathbf{z}_3 + \cdots + \alpha_{n-r} \mathbf{z}_{n-r}]_{f_j} \\ &= [\alpha_1 \mathbf{z}_1]_{f_j} + [\alpha_2 \mathbf{z}_2]_{f_j} + [\alpha_3 \mathbf{z}_3]_{f_j} + \cdots + [\alpha_{n-r} \mathbf{z}_{n-r}]_{f_j} \\ &= \alpha_1 [\mathbf{z}_1]_{f_j} + \alpha_2 [\mathbf{z}_2]_{f_j} + \alpha_3 [\mathbf{z}_3]_{f_j} + \cdots + \\ &\quad \alpha_{j-1} [\mathbf{z}_{j-1}]_{f_j} + \alpha_j [\mathbf{z}_j]_{f_j} + \alpha_{j+1} [\mathbf{z}_{j+1}]_{f_j} + \cdots + \\ &\quad \alpha_{n-r} [\mathbf{z}_{n-r}]_{f_j} \\ &= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \\ &\quad \alpha_{j-1}(0) + \alpha_j(1) + \alpha_{j+1}(0) + \cdots + \alpha_{n-r}(0) \quad \text{definition of } \mathbf{z}_j \\ &= \alpha_j \end{aligned}$$

So for all j , $1 \leq j \leq n - r$, we have $\alpha_j = 0$. Hence, the only relation of linear dependence on $S = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots, \mathbf{z}_{n-r}\}$ is the trivial one. By the definition of linear independence, the set is linearly independent, as desired. \square

Example 12.18. Find the null space of the matrix

$$A = \begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}.$$

Solution. The RREF of A is:

$$B = \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & -2 \\ 0 & \boxed{1} & 0 & -2 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The free variables are x_4 and x_5 .

Setting $x_4 = 1$ and $x_5 = 0$ gives:

$$\mathbf{z}_1 = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Setting instead $x_4 = 0$ and $x_5 = 1$ gives

$$\mathbf{z}_2 = \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Hence

$$\mathcal{N}(A) = \left\langle \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$