MATH 2010 Chapter 5

5.1 Finding Limits Using Polar Coordinates

Recall:

$$(x, y) \longleftrightarrow (r, \theta)_{pol}$$

with:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

and:

$$(x, y) = (0, 0) \Longleftrightarrow r = 0.$$

Example 5.1. Find:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}.$$

Solution.

$$= \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= \lim_{r\to 0} r\left(\cos^3\theta + \sin^3\theta\right)$$

$$= 0 \qquad \text{(Squeeze theorem)}$$

Example 5.2. Find:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + xy}{2(x^2 + y^2)}.$$

Solution.

$$= \lim_{r \to 0} \frac{r^2 \cos^2\theta + r^2 \cos\theta \sin\theta}{2r^2}$$

$$= \lim_{r \to 0} \frac{\cos^2\theta + \cos\theta \sin\theta}{2}$$

$$= \begin{cases} \frac{1}{2} & \text{if} \quad \theta = 0 \\ 0 & \text{if} \quad \theta = \frac{\pi}{2} \end{cases}$$

In other words, the function approach different values as (x, y) approaches $(0, 0)$ at different angles. Hence, the limit **does not exist** .

Example 5.3. Find:

$$\lim_{(x,y)\to(0,0)} xy \ln(x^2 + y^2).$$

Solution.

$$= \lim_{r \to 0} \underbrace{r^2 \cos\theta \sin\theta \ln(r^2)}$$

Observe that, as $r \to 0$,

$$|\cos\theta \sin\theta| \leqslant 1,$$
$$r^2 \to 0,$$
$$\ln\left(r^2\right) \to -\infty.$$

Moreover:

$$\left|r^2 \cos\theta \sin\theta \ln\left(r^2\right)\right| \leq \left|r^2 \ln\left(r^2\right)\right|$$

We have:

$$\lim_{r \to 0} r^2 \ln\left(r^2\right) = \lim_{r \to 0} \frac{\ln\left(r^2\right)}{\frac{1}{r^2}} \qquad \left(\frac{-\infty}{\infty}\right)$$

$$= \lim_{r \to 0} \frac{\frac{2r}{r^2}}{-\frac{2}{r^3}} \text{ (L' Hopital's Rule)}$$

$$= \lim_{r \to 0} -r^2 = 0$$

By Squeeze theorem, it now follows that:

$$\lim_{(x,y)\to(0,0)} xy \ln\left(x^2 + y^2\right) = 0.$$

5.2 Iterated Limits

Example 5.4. Consider:

$$f(x,y) = \frac{x+y}{x-y}$$

$$\begin{aligned}\lim_{x\to 0}\lim_{y\to 0}\frac{x+y}{x-y} &= \lim_{x\to 0}\frac{x+0}{x-0}\\ &= 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_{y\to 0}\lim_{x\to 0}\frac{x+y}{x-y} &= \lim_{y\to 0}\frac{0+y}{0-y}\\ &= -1.\end{aligned}$$

Moreover, $\lim_{(x,y)\to(0,0)}\frac{x+y}{x-y}$ does not exist (**Exercise**).

Remark.

- In general, if $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ both exist and are equal to each other, it does *NOT* follow that $\lim_{(x,y)\to(0,0)} f(x,y)$ exists.

  Counter-example:

$$f(x,y) = \begin{cases} 1 & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$$

- Conversely, if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, it also does *NOT* follow that:

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y), \quad \lim_{y\to 0}\lim_{x\to 0} f(x,y)$$

  both exist. Counter-example:

$$f(x,y) = \begin{cases} x\cos\frac{1}{y} + y\cos\frac{1}{x} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- If all three limits exist, then they are equal.

5.3 Continuity

Definition 5.5. We say that a function $f : A \longrightarrow \mathbb{R}$ in n variables is **continuous** at $\vec{a} \in A$ if:

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a}).$$

Definition 5.6. A function $\vec{f} : A \longrightarrow \mathbb{R}$ is **continuous** is f is continuous at every point in its domain A.

Example 5.7. Each "coordinate function" $f_i : \mathbb{R}^m \longrightarrow \mathbb{R}$, defined by:

$$f_i(x_1, x_2, \ldots, x_m) = x_i,$$

is continuous.

Theorem 5.8. *Let k be a scalar constant. If $f, g : A \longrightarrow \mathbb{R}$ are continuous at $\vec{a} \in A$, then:*

- *$f + g$, kf, fg are all continuous at $\vec{a}$*
- *$\dfrac{f}{g}$ is continuous at $\vec{a}$ is $g(\vec{a}) \neq 0$.*

Proof of Theorem 5.8. This follows from the properties of limits. □

Corollary 5.9. *All polynomial and rational functions (i.e. polynomial divided by another polynomial) are continuous (on their domains).*

Theorem 5.10. *If $f : A \longrightarrow \mathbb{R}$ is continuous at $\vec{a} \in A$, and $g : I \longrightarrow \mathbb{R}$ is a single-variable real-valued function continuous at $f(\vec{a})$, then $g \circ f : A \longrightarrow \mathbb{R}$ is continuous at $\vec{a}$.*

In other words:

$$\lim_{\vec{x} \to \vec{a}} g(f(\vec{x})) = g\left(\lim_{\vec{x} \to \vec{a}} f(\vec{x})\right) = g(f(\vec{a})).$$

Corollary 5.11. *Every so-called "***elementary function***" (a function constructed from constants, power functions, trigonometric, inverse trigonometric, exponential and logarithmic functions, via addition, subtraction, multiplication, division and composition) is continuous at all points in its domain.*

Example 5.12.

- Every polynomial in n variables (e.g. $f(x, y, z) = x^2yz + 5yz^2 + 16y^3 - 8$) is continuous everywhere.
- Every rational function in n variables is continuous at all points where the function is defined.
- $f(x, y) = e^{\cos(x^2+y^2)}$ is continuous at all $(x, y) \in \mathbb{R}^2$.
- $f(x, y) = \dfrac{1}{\sqrt{x^2 + y}}$ is continuous at all $(x, y) \in \mathbb{R}^2$ such that $x^2 + y > 0$.

Example 5.13. • Consider:

$$g(x,y) = \frac{x^4 - y^4}{x^2 + y^2}.$$

Since $x^2 + y^2 = 0 \Leftrightarrow (x,y) = (0,0)$, the domain of g is $\mathbb{R}^2 \backslash \{(0,0)\}$.

$$\begin{aligned}
\lim_{(x,y \to (0,0)} g(x,y) &= \lim_{(x,y) \to (0,0)} \frac{x^4 - y^4}{x^2 + y^2} \\
&= \lim_{r \to 0} \frac{r^4 \cos^4 \theta - r^4 \sin^4 \theta}{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \\
&= \lim_{r \to 0} r^2 \left(\cos^2 \theta - \sin^2 \theta \right) \\
&= 0 \quad \text{(Sandwich theorem)}
\end{aligned}$$

Hence, g *can* be extended to a continuous function on the whole $\mathbb{R}^2$ as follows:

$$g(x,y) = \begin{cases} \frac{x^4 - y^4}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

• On the other hand, consider:

$$f(x,y) = \frac{xy + y^3}{x^2 + y^2}$$

$$\begin{aligned}
\lim_{\substack{(x,y) \to (0,0) \\ y = mx}} f(x,y) &= \lim_{\substack{(x,y) \to (0,0) \\ y = mx}} \frac{xy + y^3}{x^2 + y^2} \\
&= \lim_{x \to 0} \frac{mx^2 + m^3 x^3}{x^2 + m^2 x^2} \\
&= \lim_{x \to 0} \frac{m + m^3 x}{1 + m^2} \\
&= \frac{m}{1 + m^2} = \begin{cases} 0, & \text{if } m = 0 \\ \frac{1}{2} & \text{if } m = 1 \end{cases}
\end{aligned}$$

Since the limit varies with slope, $\lim_{(x,y) \to (0,0)} f(x,y)$ does not exist.

The function f cannot be extended to a function defined on $\mathbb{R}^2$.

5.4 Partial Derivatives

Definition 5.14. Let $f : A \longrightarrow \mathbb{R}$ be a function on an open region $A \in \mathbb{R}^n$, $\vec{a} = (a_1, a_2, \ldots, a_n) \in A$. For $i = 1, 2, \ldots, n$, we define the **partial derivative**

with respect to x_i of f at $\vec{a}$ to be:

$$\frac{\partial f}{\partial x_i}(\vec{a}) = \left(\frac{d}{dx_i} f(a_1, a_2, \dots, a_{i-1}, \underbrace{x_i}_{i\text{-th coordinate}}, a_{i+1}, \dots, a_n) \right)\Bigg|_{x_i = a_i}$$
$$= \lim_{h \to 0} \frac{f(a_1, a_2, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(\vec{a})}{h}$$

Observe that as $\vec{a}$ varies, the correspondence:

$$\vec{a} \mapsto \frac{\partial f}{\partial x_i}(\vec{a})$$

defines a real-valued function on a subset A' of A consisting of those points $\vec{a} \in A$ where $\frac{\partial f}{\partial x_i}(\vec{a})$ is defined.

We have therefore a multivariable function defined as follows:

Definition 5.15.

$$\frac{\partial f}{\partial x_i} : A' \longrightarrow \mathbb{R},$$

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \dots, x_n)$$
$$= \lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{h}.$$

Notation. Other notations for $\dfrac{\partial f}{\partial x_i}$ are:

$$f_{x_i},\ \partial_i f,\ D_i f,\ \nabla_i f$$

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Example 5.16.

$$f(x, y) = x^2 + y^2$$
$$\frac{\partial f}{\partial x} = 2x + 0 = 2x \quad \text{(Regard } y \text{ as a constant)}$$
$$\frac{\partial f}{\partial y} = 0 + 2y = 2y \quad \text{(Regard } x \text{ as a constant)}$$

In particular:

$$\frac{\partial f}{\partial x}(1, -1) = 2(1) = 2 > 0$$
$$\frac{\partial f}{\partial y}(1, -1) = 2(-1) = -2 < 0$$

This means that $f(x, y)$ increases as x increases at $(1, -1)$, and it decreases as y increases at $(1, -1)$.

Example 5.17.

$$f(x,y,z) = xy^2 - \cos(xz)$$

Find f_x, f_y, f_z.

Solution.

$$f_x = y^2 + z\sin(xz)$$

$$f_y = 2xy + 0 = 2xy$$

$$f_z = 0 + x\sin(xz) = x\sin(xz)$$

Example 5.18.

$$f(x,y) = \begin{cases} 1 & \text{if } xy \geq 0; \\ 0 & \text{if } xy < 0. \end{cases}$$

Find $\dfrac{\partial f}{\partial x}(1,1), \dfrac{\partial f}{\partial x}(0,1), \dfrac{\partial f}{\partial x}(0,0)$.

Solution. $\dfrac{\partial f}{\partial x}$: Fix y, differentiate $f(x,y)$ with respect to x.

Along $y = 1$

$$f(x,1) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Hence:

$$\frac{\partial f}{\partial x}(1,1) = 0,$$

and:

$$\frac{\partial f}{\partial x}(0,1) \ \text{ DNE}$$

Along $y = 0$ We have $f(x,0) = 1$ for all $x \in \mathbb{R}$. This implies that:

$$\frac{\partial f}{\partial x}(0,0) = 0.$$

Remark. In the previous example, we can similarly conclude that: $\dfrac{\partial f}{\partial y}(0,0) = 0$.

Also, it may be shown that f is not continuous at $(0,0)$ (**exercise**).

Hence, in general, the existence of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at a point P does **not** imply that f is continuous at P.

5.5 Higher Order Partial Derivatives

Since, $\dfrac{\partial f}{\partial x_i}$ is itself a function in n variables, we can consider its partial derivative with respect to any of the variables x_j. We can likewise further consider partial derivatives of *that* partial derivative, and so on. The notation is as follows:

$$\frac{\partial^2 f}{\partial x_i^2} = f_{x_i x_i} := \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_i}\right).$$

For $j \neq i$,

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{x_i x_j} := \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right).$$

For $m \in \mathbb{N}$,

$$\frac{\partial^m f}{\partial x_i^m} = f_{\underbrace{x_i x_i \cdots x_i}_{m \text{ times}}} := \frac{\partial}{\partial x_i}\left(\frac{\partial^{m-1} f}{\partial x_i^{m-1}}\right)$$

For $i_1, i_2, \ldots, i_m \in \{1, 2, 3, \ldots, n\}$,

$$\frac{\partial^m f}{\partial x_{i_m} \partial x_{i_{m-1}} \partial x_{i_{m-2}} \cdots \partial x_{i_1}} = f_{x_{i_1} x_{i_2} \cdots x_{i_m}} := \frac{\partial}{\partial x_{i_m}}\left(\frac{\partial^{m-1} f}{\partial x_{i_{m-1}} \partial x_{i_{m-2}} \cdots \partial x_{i_1}}\right).$$

Example 5.19. Find all first and second order partial derivatives of:

$$f(x, y) = x \sin y + y^2 e^{2x}$$

Solution.

$$f_x = \sin y + 2y^2 e^{2x}$$

$$f_y = x \cos y + 2y e^{2x}$$

$$f_{xx} = (f_x)_x = 4y^2 e^{2x}$$

$$f_{xy} = (f_x)_y = \cos y + 4y e^{2x}$$

$$f_{yx} = (f_y)_x = \cos y + 4y e^{2x}$$

$$f_{yy} = (f_y)_y = -x \sin y + 2e^{2x}$$

Is $f_{xy} = f_{yx}$ a coincidence?

Example 5.20. Compute $f_{xy}(0,0), f_{yx}(0,0)$, where:

$$f(x,y) = \begin{cases} \dfrac{xy\left(x^2-y^2\right)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution. By definition, $f_{xy} = \left(f_x\right)_y$.

So, $f_{xy}(0,0) = \lim_{k\to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

Need to find: $f_x(0,k)$ for $k \neq 0$ and $f_x(0,0)$ for $k \neq 0$,

$f = \dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}$ near $(0,k)$.

$$f_x = \frac{\left(x^2+y^2\right)\left(3x^2y-y^3\right) - xy\left(x^2-y^2\right)(2x)}{\left(x^2+y^2\right)^2}$$

near $(0,k)$

Hence:

$$\begin{aligned} f_x(0,k) &= \frac{k^2\left(-k^3\right) - 0}{k^4} = -k \\ f_x(0,0) &= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0-0}{h} = 0 \\ f_{xy}(0,0) &= \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \\ &= \lim_{k\to 0} \frac{-k-0}{k} = -1 \end{aligned}$$

Similar calculation gives: $f_{yx}(0,0) = 1$.

(Alternatively, note that $f(x,y) = -f(y,x)$. Hence $f_{yx}(0,0) = -f_{xy}(0,0) = 1$.)

Hence, in this example, $f_{xy}(0,0) \neq f_{yx}(0,0)$

Theorem 5.21 (Mixed Derivative Theorem)**.** *Let x and y be two of the variables of a real-valued function f in multiple variables. If f_{xy} and f_{yx} exist and are continuous on an open region containing a point $\vec{a}$, then:*

$$f_{xy}(\vec{a}) = f_{yx}(\vec{a}).$$

Proof of Mixed Derivative TheoremClairaut's Theorem. We prove the theorem for the special case where $f : A \longrightarrow \mathbb{R}$ has two variables (i.e. $A \subseteq \mathbb{R}^2$).

Without loss of generality, we may assume that $\vec{a} = (0,0) \in A$. We want to show that:

$$f_{xy}(0,0) = f_{yx}(0,0)$$

Let h, k be any positive real numbers such that $[0, h] \times [0, k] \subseteq A$. Let:

$$\alpha = (f(h, k) - f(h, 0)) - (f(0, k) - f(0, 0))$$

Let:

$$g(x) = f(x, k) - f(x, 0), \quad 0 \leq x \leq h.$$

Then:

$$\alpha = g(h) - g(0),$$

and:

$$g'(x) = f_x(x, k) - f_x(x, 0).$$

By the Mean Value Theorem , there exists $h_1 \in (0, h)$ such that:

$$\frac{\alpha}{h} = \frac{g(h) - g(0)}{h} = g'(h_1) = f_x(h_1, k) - f_x(h_1, 0).$$

By MVT again, there exists $k_1 \in (0, k)$ such that:

$$\frac{f_x(h_1, k) - f_x(h_1, 0)}{k} = f_{xy}(h_1, k_1).$$

Hence:

$$\alpha = h\,[f_x(h, k) - f_x(h, 0)] = hkf_{xy}(h_1, k_1).$$

Similarly, there exists $(h_2, k_2) \in (0, h) \times (0, k)$ such that:

$$\alpha = hkf_{yx}(h_2, k_2)$$

Hence, for any positive real numbers h, k sufficiently small, we have:

$$f_{xy}(h_1, k_1) = f_{yx}(h_2, k_2) \tag{5.1}$$

for some $(h_1, k_1), (h_2, k_2)$ lying the rectangle $[0, h] \times [0, k]$.

If we let $(h, k) \to (0, 0)$, then $(h_1, k_1), (h_2, k_2) \to (0, 0)$. So, from an intuitive perspective, it follows from (5.1), and the continuity of f_{xy} and f_{yx} at $(0, 0)$, that:

$$f_{xy}(0, 0) = f_{yx}(0, 0).$$

More rigorously:

Suppose $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Then, $d := |f_{xy}(0, 0) - f_{yx}(0, 0)| > 0$.

The continuity of f_{xy} and f_{yx} at $(0,0)$ implies that there exists $\delta > 0$ such that for all $(x,y) \in B_\delta(0,0)$, we have:

$$|f_{xy}(x,y) - f_{xy}(0,0)| < d/2$$

and

$$|f_{yx}(x,y) - f_{yx}(0,0)| < d/2.$$

Hence, if we take (h,k) such that $0 < \|(h,k)\| < \delta$, then, (5.1) implies that the intervals:

$$(f_{xy}(0,0) - d/2, f_{xy}(0,0) + d/2), \quad (f_{yx}(0,0) - d/2, f_{yx}(0,0) + d/2),$$

have nonempty intersection (i.e. the common value $f_{xy}(h_1,k_1) = f_{yx}(h_2,k_2)$ lies in both intervals). This contradicts the assumption that the distance d between $f_{xy}(0,0)$ and $f_{yx}(0,0)$ is nonzero. □