MATH 2010 Chapter 4

4.1 Vector-Valued Functions in Multiple Variables

Let

$$\vec{f} : \Omega \longrightarrow \mathbb{R}^m,$$

be a vector-valued function, where $\Omega \subseteq \mathbb{R}^n$.

Definition 4.1. The **graph** of $\vec{f}$ is:

$$\text{Graph}(\vec{f}) = \left\{ \left(\vec{x}, \vec{f}(\vec{x}) \right) : \vec{x} \in \Omega \right\} \subseteq \mathbb{R}^{n+m}$$

4.1.1 Level Set

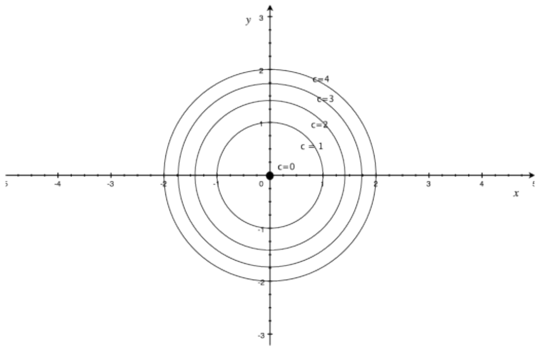
For a function $\vec{f} : \Omega \longrightarrow \mathbb{R}^m$, $\Omega \subseteq \mathbb{R}^n$, in n variables, and $\vec{c} \in \mathbb{R}^m$, the **level set** of $\vec{f}$ corresponding to $\vec{c}$ is the set of points $(x_1, x_2, \ldots, x_n) \in \Omega$ such that

$$\vec{f}(x_1, x_2, \ldots, x_n) = \vec{c}$$

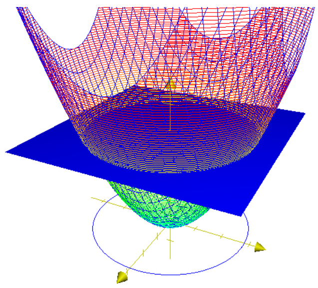
- If $n = 2$, then a level set of $\vec{f}$ is typically a curve in the xy-plane, and is often called a **level curve**.
- If $n = 3$, then a level set is typically a surface in the xyz-space, and is often called a **level surface**.

Example 4.2. $f(x, y) = x^2 + y^2$.

- For $c = -2, -1$, the level sets $f(x, y) = x^2 + y^2 = c$ are empty.
- For $c = 0$, the level set $f(x, y) = x^2 + y^2 = 0$ consists of the single point $(0, 0)$.
- For $c > 0$, the level set $f(x, y) = x^2 + y^2 = c$ is the circle in $\mathbb{R}^2$ centred at the origin with radius $\sqrt{c}$.



Each level set $f(x, y) = c$ corresponds to (the projection onto the xy-plane of) the intersection of the surface $z = f(x, y)$ and the horizontal (hence "level") plane $z = c$:



IFRAME

4.2 Limits of Multivariable Functions

First, recall Closure.

Definition 4.3 (Limit)**.** Let $\vec{f} : A \longrightarrow \mathbb{R}^m$ be a vector-valued function on $A \subseteq \mathbb{R}^n$.

For any $\vec{a} \in \bar{A}$, we say that: *The limit of $\vec{f}$ at $\vec{a}$ is $\vec{L}$*

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$$

if: For all $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$\|\vec{f}(\vec{x}) - \vec{L}\| < \varepsilon$$

for all $\vec{x} \in A$ which satisfies $0 < \|\vec{x} - \vec{a}\| < \delta$.

Example 4.4. Let:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R},$$
$$f(x, y) = x + y, \quad (x, y) \in \mathbb{R}^2.$$

Then,

$$\lim_{(x,y)\to(1,2)} f(x, y) = 3.$$

Proof of Example 4.4. Show that given any $\varepsilon > 0$, one can find $\delta > 0$ such that if $0 < \|(x, y) - (1, 2)\| < \delta$, then $|f(x, y) - 3| < \varepsilon$.

Idea:

$$\begin{aligned} |f(x, y) - 3| &= |(x - 1) + (y - 2)| \\ &\leqslant |x - 1| + |y - 2| \\ \|(x, y) - (1, 2)\| &= \sqrt{(x - 1)^2 + (y - 2)^2}. \end{aligned}$$

For example, for $\varepsilon = 1$, one can pick $\delta = \frac{1}{2}$:

If $\|(x, y) - (1, 2)\| < \delta = \frac{1}{2}$, then:

$$|x - 1| = \sqrt{(x - 1)^2} \leqslant \sqrt{(x - 1)^2 + (y - 2)^2} < \frac{1}{2}$$
$$|y - 2| = \sqrt{(y - 2)^2} \leqslant \sqrt{(x - 1)^2 + (y - 2)^2} < \frac{1}{2}$$

This implies that:

$$|f(x, y) - 3| \leqslant |x - 1| + |y - 2| < \frac{1}{2} + \frac{1}{2} = 1 = \varepsilon$$

Similarly, for $\varepsilon = \frac{1}{100}$, one can pick $\delta = \frac{1}{200}$

In general, we need to do it for any $\varepsilon > 0$ For any given $\varepsilon > 0$, one can pick $\delta = \frac{\varepsilon}{2}$. Then:

$$\begin{aligned} &\|(x, y) - (1, 2)\| < \delta = \frac{\varepsilon}{2} \\ \Rightarrow &|f(x, y) - 3| = |x + y - 3| \leqslant |x - 1| + |y - 2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence, $\lim_{(x,y)\to(1,2)} f(x, y) = 3.$ □

Example 4.5. Let:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R},$$
$$f(x, y) = x^2 + y^2, \quad (x, y) \in \mathbb{R}^2.$$

Then,

$$\lim_{(x,y)\to(0,0)} f(x, y) = 0.$$

Proof of Example 4.5. For all $\varepsilon > 0$, we need to find $\delta > 0$ such that:
if:

$$0 < \|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta,$$

then:

$$|f(x, y) - 0\| = \left|x^2 + y^2\right| < \varepsilon.$$

Exercise: Complete the rest of the proof. □

Proposition 4.6. *Let $A \subseteq \mathbb{R}^n$, $a \in A$, $\vec{f} : A \longrightarrow \mathbb{R}^m$, where:*

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}, \quad f_i : A \longrightarrow \mathbb{R}.$$

Then,

$$\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix}$$

if and only if

$$\lim_{\vec{x} \to \vec{a}} f_i(\vec{x}) = l_i$$

for $i = 1, 2, \ldots, m$.

Example 4.7. Let:

$$\vec{f} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\vec{f}(x, y) = \begin{bmatrix} x + y \\ x^2 + y^2 + 1 \end{bmatrix}, \quad (x, y) \in \mathbb{R}^2.$$

Then,

$$\lim_{(x,y) \to (1,2)} \vec{f}(x, y) = \begin{bmatrix} \lim_{(x,y)\to(1,2)} x + y \\ \lim_{(x,y)\to(1,2)} x^2 + y^2 + 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Proposition 4.8. *Let $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function. Let $\gamma, \psi : \mathbb{R} \longrightarrow \mathbb{R}^n$ be the parameterization of two paths in $\mathbb{R}^n$, with $\gamma(0) = \psi(0) = \vec{a}$. If $\lim_{t\to 0} \vec{f}(\gamma(t))$ or $\lim_{t\to 0} \vec{f}(\psi(t))$ does not exist, or the two limits are not equal to each other, then the limit $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ does not exist.*

In fact:

Theorem 4.9. $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L}$ *if and only if the limit of* $\vec{f}(\vec{x})$ *at* $\vec{a}$ *along any path through* $\vec{a}$ *exists and is equal to* $\vec{L}$.

Example 4.10. Consider $\lim_{(x,y)\to(0,0)} f(x,y)$, where:

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

Let:

$$\gamma(t) = (t,t), \quad t \in \mathbb{R},$$
$$\psi(t) = (t,-t), \quad t \in \mathbb{R}.$$

Then,

$$\gamma(0) = \psi(0) = (0,0),$$

and:

$$\lim_{t\to 0} f(\gamma(t)) = \lim_{t\to 0} \frac{t\cdot t}{t^2+t^2} = \lim_{t\to 0} \frac{t^2}{2t^2} = \frac{1}{2},$$
$$\lim_{t\to 0} f(\psi(t)) = \lim_{t\to 0} \frac{t\cdot(-t)}{t^2+(-t)^2} = \lim_{t\to 0} -\frac{t^2}{2t^2} = -\frac{1}{2},$$

Since $\lim_{t\to 0} f(\gamma(t)) \neq \lim_{t\to 0} f(\psi(t))$, we conclude that the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Remark. Let $\vec{a} = (x_0, y_0)$. If $\lim_{x\to x_0} f(x,y_0) = \lim_{x\to y_0} f(x_0,y) = L$, it is not necessarily true that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, or that the limit even exists.

Example 4.11.

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$f(x,y) = \begin{cases} 1 & \text{if} \quad 0 < y < x^2 \\ 0 & \text{otherwise} \end{cases}$$

Find $\lim_{(x,y)\to\vec{a}} f(x,y)$, where:

1. $\vec{a} = (0,1)$
2. $\vec{a} = (1,1)$
3. $\vec{a} = (0,0)$

4.2.1 Proporties of Limits

If all limits on the right-hand side exists, then the limit of the left-hand side exists and the formula holds:

1. $\lim\limits_{\vec{x}\to\vec{a}}(\vec{f}(x) \pm \vec{g}(x)) = \lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) \pm \lim\limits_{\vec{x}\to\vec{a}} \vec{g}(\vec{x})$.

2. $\lim\limits_{\vec{x}\to\vec{a}} k\vec{f}(\vec{x}) = k \lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})$ for any scalar constant k.

3. If $\vec{f}$ and $\vec{g}$ are real-valued, then $\lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})\vec{g}(\vec{x}) = \left(\lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})\right) \left(\lim\limits_{\vec{x}\to\vec{a}} \vec{g}(\vec{x})\right)$.

4. $\lim\limits_{\vec{x}\to\vec{a}} \dfrac{f(\vec{x})}{g(\vec{x})} = \dfrac{\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x})}{\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x})}$ provided that $\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) \neq 0$.

5. $$\lim_{\vec{x}\to\vec{a}}(f(\vec{x}))^n = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^n \qquad \text{for all } n \in \mathbb{N} = \{1, 2, 3, \ldots\},$$

6. $$\lim_{\vec{x}\to\vec{a}} (f(x))^{1/n} = \left(\lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)^{1/n} \qquad \text{for all odd positive integers } n.$$

7. If $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L > 0$, then
$$\lim_{\vec{x}\to\vec{a}} (f(\vec{x}))^{1/n} = L^{1/n}$$
for all $n \in \mathbb{N}$.

Theorem 4.12 (Squeeze Theorem)**.** *Let $f, g, h : \Omega \longrightarrow \mathbb{R}$ be real-valued functions on $\Omega \in \mathbb{R}^n$.*

If:
$$g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x})$$
for all $\vec{x}$ near $\vec{a} \in \Omega$, and
$$\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L,$$
then:
$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L.$$

Corollary 4.13. *If $|f(\vec{x})| \leq g(\vec{x})$ near $\vec{a}$ and $\lim\limits_{\vec{x}\to\vec{a}} g(\vec{x}) = 0$, then $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = 0$.*

Example 4.14. Find:

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right).$$

Solution. Note:

$$\left|\cos\left(\frac{1}{x^2+y^2}\right)\right| \leqslant 1 \Rightarrow \left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| \leqslant |x|$$

Also,

$$\lim_{(x,y)\to(0,0)} |x| = 0.$$

Hence,

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right) = 0$$

by the Squeeze Theorem.

Example 4.15. Find:

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2}.$$

Solution. Note:

$$\begin{aligned}\left|\frac{(x-1)^2\ln x}{(x-1)^2+y^2}\right| &= \left|\frac{(x-1)^2}{(x-1)^2+y^2}\right| \cdot |\ln x| \\ &\leqslant |\ln x|\end{aligned}$$

Also, $\lim_{(x,y)\to(1,0)} |\ln x| = |\ln(1)| = 0$
By squeeze theorem,

$$\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{(x-1)^2+y^2} = 0$$

Remark. If $a \geqslant b$, then
$ca \leqslant cb$ if $c > 0$
$ca \leqslant cb$ if $c < 0$