MATH 2010 Chapter 3

3.1 Polar Coordinates in $\mathbb{R}^2$

A point $P = (x, y) \in \mathbb{R}^2$ can be represented by:

$$r = \sqrt{x^2 + y^2} = \text{ distance from origin.}$$

$$\theta = \text{ angel from the positive } x- \text{ axis to } \overrightarrow{OP} \text{ in counter-clockwise direction.}$$

If $x, y > 0$, then we can take $\theta = \arctan\left(\frac{y}{x}\right)$.

The angle formula above needs to be adjusted for points in other qudrants. For example, if $x < 0, y > 0$ (Quadrant II), then:

$$\theta = \pi + \arctan\left(\frac{y}{x}\right)$$

Remark.

- For $P = (0, 0)$, we have $r = 0$, but θ is not (uniquely) defined.
- Different conventions for ranges of r and θ:

$$r \in [0, \infty) \text{ or } \mathbb{R}$$

$$\theta \in [0, 2\pi) \text{ or } \mathbb{R}$$

In this course, we usually take:

$$r \in [0, \infty), \quad \theta \in \mathbb{R}.$$

3.1.1 Change of Coordinates Fomula

If the polar coordinates for a point (x, y) is (r, θ), then:

$$\begin{cases} x = r\cos\theta; \\ y = r\sin\theta. \end{cases}$$

3.1.2 Curves in Polar Coodinates

Example 3.1 (Circle with radius r0)**. Polar equation**

$$r = r_0$$

Parametric form

$$\begin{cases} r = r_0 \\ \theta = t, \quad t \in [0, 2\pi]. \end{cases}$$

Example 3.2 (Half ray from origin)**. Polar equation**

$$\theta = \theta_0$$

Polar equation

$$\begin{cases} r = t, \quad t \in [0, \infty) \\ \theta = \theta_0. \end{cases}$$

Example 3.3 (Archimedes Spiral)**.** Let $k > 0$ be a constant

Polar equation

$$r = k\theta$$

Polar equation

$$\begin{cases} r = kt, t \in [0, \infty) \\ \theta = t, t \in [0, \infty) \end{cases}$$

Example 3.4.

$$r = 4\cos\theta$$

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Observe that the origin, corresponding to $r = 0, \theta = \pi/2$, lies on the graph of $r = 4\cos\theta$. Hence, the solution set of $r = 4\cos\theta$ is equal to the solution set of:

$$r^2 = 4r\cos\theta,$$

which is equivalent to the Cartesian equation:

$$x^2 + y^2 = 4x$$

Completing the square, the equation above is equivalent to:

$$(x - 2)^2 + y^2 = 2^2,$$

which corresponds to the circle of radius 2 centered at $(2, 0)$.

Example 3.5.

$$r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}.$$

(Hint: The graph is a straight line in the Cartesian plane.)

Example 3.6. IFRAME

It is sometimes convenient to allow $r < 0$ in polar coordinates.

For instance, to describe a line through the origin which forms an angle of $\pi/6$ with the positive x-axis, we can simply describe it as the graph of:

$$\theta = \pi/6$$

with the assumption that $r \in \mathbb{R}$.

(If we only let $r \geq 0$, then we only get "half" a line.)

Example 3.7. Let $a > 1$ be constant. Consider:

$$r = 1 - acos\theta$$

If we require that $r \geq 0$, then the equatio above only possibly holds for $\theta \in [\delta, 2\pi - \delta]$, where $\delta = \arccos(1/a)$.

IFRAME

On the other hand, of we let allow r to also be negative, then for any $\theta \in [0, 2\pi]$ there is an r for which the equation holds. The resulting graph would have one extra "loop".

IFRAME

3.2 Coordinate Systems in $\mathbb{R}^3$

Definition 3.8. Given a point $P \in \mathbb{R}^3$ with Cartesian coordinates (x, y, z).

The **cylindrical coordinates** of P is:

$$(r, \theta, z),$$

where (r, θ) are the polar coordinates of (x, y).

Hence,

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned}$$

IFRAME

Example 3.9. Let $a, b \in \mathbb{R}$. A vertical helix with radius a may be described with cylindrical coordinates as follows:

$$\begin{cases} r = a \\ \theta = t \\ z = bt \end{cases}, \quad t \in [0, 2\pi]$$

Definition 3.10. Given a point $P \in \mathbb{R}^3$ with Cartesian coordinates (x, y, z).

The **spherical coordinates** of P is:

$$(\rho, \theta, \phi),$$

where:

- $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance between P and the origin.
- θ is the angle coordinate of the polar coordinates of (x, y) in the xy-plane.
- ϕ is the angle between the positive z-axis and $\overrightarrow{OP}$.

Hence,

$$\begin{aligned} x &= \rho \sin \phi \cos \theta, \\ y &= \rho \sin \phi \sin \theta, \\ z &= \rho \cos \phi. \end{aligned}$$

IFRAME

Example 3.11 (Sphere)**.**

$$\rho = 2.$$

Example 3.12 (Cone)**.**

$$\phi = \pi/4.$$

Example 3.13 (Half Plane)**.**

$$\theta = \pi/3.$$

Example 3.14 (Circle)**. Equations:**

$$\begin{cases} \rho = 3, \\ \phi = \pi/2. \end{cases}$$

Parametric Form:

$$(\rho, \theta, \phi)_{sph} = (3, t, \pi/2), \quad t \in [0, 2\pi].$$

3.3 Topological Terminology

Let $\vec{x}_0 \in \mathbb{R}^n$, $\varepsilon > 0$.

Definition 3.15. The **open ball** with radius ε centered at $\vec{x}_0$ is:

$$B_\varepsilon(\vec{x}_0) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| < \varepsilon.\}$$

The **closed ball** with radius ε centered at $\vec{x}_0$ is:

$$\overline{B_\varepsilon(\vec{x}_0)} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{x}_0\| \leq \varepsilon.\}$$

Let $S \subseteq \mathbb{R}^n$.

Definition 3.16.

- The **interior** of S is the set:

$$\text{Int}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(\vec{x}) \subset S \text{ for some } \varepsilon > 0.\}$$

  Points in $\text{Int}(S)$ are called **interior points** of S.

- The **exterior** of S is the set:

$$\text{Ext}(S) = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \backslash S \text{ for some } \varepsilon > 0.\}$$

  Points in $\text{Ext}(S)$ are called **exterior points** of S.

- The **boundary** of S is the set:

$$\partial S = \{\vec{x} \in \mathbb{R}^n : B_\varepsilon(\vec{x}) \cap S \neq \varnothing \text{ and } B_\varepsilon(\vec{x}) \cap \mathbb{R}^n \backslash S \neq \varnothing, \text{ for all } \varepsilon > 0.\}$$

  Points in $\partial(S)$ are called **boundary points** of S.

IMAGE

Example 3.17.

$$S = \left\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\right\} \subseteq \mathbb{R}^2$$

Proposition 3.18. *Let $S \subseteq \mathbb{R}^n$. Then,*

- $\mathbb{R}^n$ *is the disjoint union of* $\text{Int}(S), \text{Ext}(S)$ *and* ∂S.
- $\text{Int}(S) \subseteq S$, $\text{Ext}(S) \subseteq \mathbb{R}^n \backslash S$.

Definition 3.19. A subset $S \subseteq \mathbb{R}^n$ is said to be

- **open** if for all $x \in S$, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.

- **closed** if $\mathbb{R}^n \backslash S$ is open.

Definition 3.20 (Closure)**.** The **closure** of a set $A \subseteq \mathbb{R}^n$ is:

$$\bar{A} = A \cup \partial A$$

Remark. The closure of any set is always closed.

Theorem 3.21. *A subset $S \subseteq \mathbb{R}^n$ is:*

- **open** *if and only if $S = \text{Int}(S)$.*
- **closed** *if and only if $S = \text{Int}(S) \cup \partial S$.*

Example 3.22.

Subset $S \subseteq \mathbb{R}^n$	$B_1(0,0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$	$\overline{B_1(0,0)} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$
$\text{Int}(S)$		
$\text{Ext}(S)$		
∂S		
Open?		
Closed?		

Remark.

- There are exactly two subsets of $\mathbb{R}^n$ which are both open and closed:

$$\mathbb{R}^n, \varnothing$$

- Some subsets of $\mathbb{R}^n$ are neither open nor closed:

$$\{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 4\} \subseteq \mathbb{R}^n$$

$$(0,1] \subseteq \mathbb{R}$$

$$\mathbb{Q} \subseteq \mathbb{R}$$

Exercise : $\partial \mathbb{Q} = \mathbb{R}$.

Definition 3.23. A subset $S \subseteq \mathbb{R}^n$ is said to be:

- **bounded** if there exists $M > 0$ such that:

$$S \subseteq B_M(\vec{0}) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| < M\}$$

- **unbounded** if it is not bounded.

Definition 3.24. A subset $S \subseteq \mathbb{R}^n$ is said to be **path-connected** if any two points in S can be connected by a curve in S.

Theorem 3.25 (Jordan Curve Theorem)**.** *A simple closed curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two path-connected components, with one bounded and one unbounded.*