MATH 2010 Chapter 12

Question:

When does f have global extrema subject to constraint $g = c$?

A sufficient condition:

- The level set $S = \{g = c\}$ is closed and bounded
- f is continuous on S

By EVT, f has global extrema on S.

12.1 Quadratic Constraint on 2 Variables (Conic Section)

$$g(x, y) = Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F$$

Some typical examples of $g = c$:

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a, b > 0$ (Ellipse. Circle if $a = b$)

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2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, a, b > 0$ (Hyperbola)

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   Remark. $xy = c, c \neq 0$ also a hyperbola.

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3. $y = ax^2, a \neq 0$ (Parabola) (only 1 quadratic term)

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4. Degenerate Cases

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0 \rightsquigarrow$ a point $(0,0)$
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1 \rightsquigarrow$ empty set
- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \rightsquigarrow \frac{x}{a} = \pm\frac{y}{b}$
  (a pair of intersecting lines)
- $x^2 = c \rightsquigarrow x = \pm\sqrt{c}$
  (a pair of parallel lines (double line if $c = 0$))

By a change of coordinates, any quadratic constraint $g(x, y) = c$ can be transformed to one of the forms above:
$\Rightarrow$ Ellipse, Hyperbola, Parabola, Degenerate
Each quadratic constraint corresponds to the intersection of a plane with a cone:
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Example 12.1.

$$17x^2 - 12xy + 8y^2 + 16\sqrt{5}x - 8\sqrt{5}y = 0$$

$$\Leftrightarrow \frac{(u+1)^2}{1^2} + \frac{v^2}{2^2} = 1, \text{ where } u = \frac{2x - y}{\sqrt{5}} \quad v = \frac{x + 2y}{\sqrt{5}}$$

Remark. In the last example, u and v are chosen so that u-axis $\perp$ v-axis.

Such u and v can be found using theory of symmetric matrices in linear algebra. Among the non-degenerate quadratic constraints above, only ellipse is closed and bounded.

Any continuous $f(x, y)$ restricted to an ellipse has both global maximum and global minimum.

It is not true for hyperbola and parabola:

A continuous $f(x, y)$ restricted to a hyperbola or parabola may not have global maximum or minimum.

12.2 Quadratic Constraint for 3-variable

$$g(x, y, z) = Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rzx + Dx + Ey + Fz + G$$

12.2.1 Some typical examples of $g = c$

Graph $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, a, b, c > 0$

How to graph it?

Start with the unit sphere:

$$x^2 + y^2 + z^2 = 1$$

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Then, stretch in x, y, z directions according to the values of a, b, c:

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Graph $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Up to rescaling, can assume $a = b = c = 1$

$$\rightsquigarrow x^2 + y^2 - z^2 = 1$$

Let $r = \sqrt{x^2 + y^2}$ = distance from (x, y, z) to z-axis $r^2 - z^2 = 1$ Hyperbola

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$x^2 + y^2 - z^2 = 1 \cdots (2)$

Hyperboloid of 1 sheet

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Graph $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

$r^2 - z^2 = -1$ Hyperbola

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$x^2 + y^2 - z^2 = -1$ Hyperboloid of 2 sheets

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Exercise 12.2. Graph

- $x^2 + y^2 - z^2 = 0$ (Elliptical cone)
- $z = x^2 + y^2$ (Elliptical Paraboloid)
- $z = x^2 - y^2$ (Hyperbolic Paraboloid)

12.2.2 Graph of standard quadratic surfaces

Example 12.3.

$x^2 + y^2 = 1$ Cylinder of Ellipse

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$z = x^2$ Cylinder of parabola

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Similar to the case of 2-variable:

Any quadratic constraint $g(x, y, z) = c$ can be transformed to one of the standard forms by a change of coordinates. Among the cases above, only ellipsoid is closed and bounded.

Any continuous $f(x, y)$ restricted to an ellipsoid has both global maximum and global minimum.

This is not the case for other quadratic surfaces.

Back to finding maximum/minimum under constraint.

Example 12.4. Find the point on the ellipse:

$$x^2 + xy + y^2 = 9$$

(**Exercise.** Show that this is indeed an ellipse.) with maximum x-coordinate.

Solution. Let $f(x, y) = x$

$$g(x, y) = x^2 + xy + y^2$$

Maximize f under constraint $g = 9$

The ellipse $g = 9$ is closed and bounded.

f is continuous. By EVT, maximum exists.

$$\nabla f = [1 \quad 0]$$

$$\nabla g = [2x + y \quad x + 2y]$$

Note $\nabla g = [0 \quad 0] \Leftrightarrow (x, y) = (0, 0)$

$(0, 0)$ is not on the ellipse. Use Lagrange Multipliers,

$$\left\{ \begin{array}{rcl} \nabla f & = & \lambda \nabla g \\ g & = & 9 \end{array} \right. \Rightarrow \left\{ \begin{array}{ccll} 1 & = & \lambda(2x + y) & \cdots (i) \\ 0 & = & \lambda(x + 2y) & \cdots (ii) \\ x^2 + xy + y^2 & = & 9 & \cdots (iii) \end{array} \right.$$

$(i) \Rightarrow \lambda \neq 0$

$\therefore (ii) \Rightarrow x + 2y = 0 \Rightarrow x = -2y \cdots (iv)$

Put (iv) into (iii) ,

$$(-2y)^2 + (-2y)y + y^2 = 9 \Rightarrow 3y^2 = 9, y = \pm\sqrt{3}$$

By (iv) , $(x, y) = (-2\sqrt{3}, \sqrt{3})$ or $(2\sqrt{3}, -\sqrt{3})$

Comparing x-coordinates, answer is $(2\sqrt{3}, -\sqrt{3})$.

Example 12.5. Find the point(s) on the hyperboloid $xy - yz - zx = 3$ closest to the origin.

Remark. It may be shown such closest point(s) exist.

For example, after a suitable change of coordinates, the surface is equivalent to the "two-piece" hyperboloid:

$$x^2 + y^2 - z^2 = -1$$

The distance between the origin and any point (x, y, z) on the hyperboloid above is simply:

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{2x^2 + 2y^2 + 1} \geq 1,$$

which clearly has an absolute minimum.

However, the hyperboloid is not bounded ⇒ farthest point does not exist.

Solution. Let $f(x, y, z) = \|(x, y, z) - (0, 0, 0)\|^2 = x^2 + y^2 + z^2$

Minimize f under constraint

$$g(x, y, z) = xy - yz - zx = 3$$

$$\nabla f = [2x \quad 2y \quad 2z] \quad \nabla g = [y - z \quad x - z \quad -x - y]$$

Note $\nabla g \neq [0, 0, 0]$ on $g = 3$

Use Lagrange Multipliers,

$$\begin{cases} \nabla f &= \lambda \nabla g \\ g &= 3 \end{cases} \underset{\text{(Ex)}}{\Rightarrow} \begin{cases} (x, y, z) &= \pm(1, 1, -1) \\ \lambda &= 1 \end{cases}$$

Note $f(1, 1, -1) = f(-1, -1, 1) = 3$

$\therefore$ Closest points are $\pm(1, 1, -1)$

Corresponding distance $= \sqrt{3}$

12.3 Lagrange Multipliers - Multiple Constraints

Theorem 12.6. *Lagrange Multipliers with multiple constraints*

Let $f, g_1, g_2, \cdots, g_k$ be C^1 functions on $\Omega \subseteq \mathbb{R}^n$

$$S = \{x \in \Omega : g_i(x) = c_i, i = 1, \cdots, k\}$$

Suppose

1. *a is a local extremum of f on S*

2. *$\nabla g_1(a), \cdots, \nabla g_k(a)$ are linearly independent*

Then

$$\begin{cases} \nabla f(a) & = & \sum_{i=1}^{k} \lambda_i \nabla g_i(a) & \textit{for some } \lambda_1, \cdots, \lambda_k \in \mathbb{R} \\ g_i(a) & = & c_i & \textit{for } i = 1, \cdots, k \end{cases}$$

Example 12.7. Maximize $f(x, y, z) = x^2+2y-z^2$ on the line $L = \begin{cases} 2x - y & = & 0 \\ y + z & = & 0 \end{cases}$ in $\mathbb{R}^3$

It is given that f has maximum on L

Solution. Let $g_1(x, y, z) = (2x - y)$ and $g_2(x, y, z) = y + z$

$$\nabla f = [2x \quad 2 \quad -2z]$$

$$\left.\begin{array}{l} \nabla g_1 = [2 \quad -1 \quad 0] \\ \nabla g_2 = [0 \quad 1 \quad 1] \end{array}\right\} \text{ linearly independent}$$

Use Lagrange Multipliers,

$$\begin{cases} \nabla f & = & \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 & = & 0 \\ g_2 & = & 0 \end{cases}$$

$$\text{Hence } \begin{cases} 2x & = & 2\lambda_1 + 0\lambda_2 & \cdots (1) \\ 2 & = & -\lambda_1 + \lambda_2 & \cdots (2) \\ -2z & = & 0\lambda_1 + \lambda_2 & \cdots (3) \\ 2x - y & = & 0 & \cdots (4) \\ y + z & = & 0 & \cdots (5) \end{cases}$$

$(4), (5) \Rightarrow 2x = y = -z$

$(1), (3) \Rightarrow \lambda_1 = x \quad \lambda_2 = -2z$

$(2) \Rightarrow -x - 2z = 2 \Rightarrow -x + 4x = 2 \Rightarrow x = \frac{2}{3}$

$\Rightarrow y = \frac{4}{3}, z = -\frac{4}{3}$ Since solution is unique and maximum exists, it must occur at $(x, y, z) = (\frac{2}{3}, \frac{4}{3}, -\frac{4}{3})$ with maximum value $f(\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}) = \frac{4}{3}$

Example 12.8. Find the minimum distance (provided that it exists) between

$$C : xy = 1 \text{ and } L : x + 4y = \frac{15}{8}$$

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Solution. Let $f(x,y,u,v) = \|(x,y)-(u,v)\|^2 = (x-u)^2+(y-v)^2$

To find distance:

Minimize $f(x,y,u,v)$ under constraints

$$g_1(x,y,u,v) = xy = 1$$

$$g_2(x,y,u,v) = u+4v = \frac{15}{8}$$

$$\begin{aligned} \nabla f &= [2(x-u) \quad 2(y-v) \quad -2(x-u) \quad -2(y-u)] \\ \nabla g_1 &= [y \quad x \quad 0 \quad 0] \\ \nabla g_2 &= [0 \quad 0 \quad 1 \quad 4] \end{aligned}$$

$\nabla g_1, \nabla g_2$ are linearly independent $\Leftrightarrow (x,y) \neq (0,0)$

But $xy = 1 \Rightarrow \nabla g_1, \nabla g_2$ are linearly independent on $g_1 = 1$ and $g_2 = \frac{15}{8}$

Use Lagrange Multipliers,

$$\begin{cases} \nabla f &= \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \\ g_1 &= 1 \\ g_2 &= \frac{15}{8} \end{cases} \Rightarrow \begin{cases} 2(x-u) &= \lambda_1 y & \cdots & (1) \\ 2(y-v) &= \lambda_1 x & \cdots & (2) \\ -2(x-u) &= \lambda_2 & \cdots & (3) \\ -2(y-v) &= 4\lambda_2 & \cdots & (4) \\ xy &= 1 & \cdots & (5) \\ u+4v &= \frac{15}{8} & \cdots & (6) \end{cases}$$

Case 1:

If $\lambda_1 = 0$ or $\lambda_2 = 0$, then

$$x = u, y = v$$

(6) $\Rightarrow x = \frac{15}{8} - 4y$

(5) $\Rightarrow (\frac{15}{8} - 4y)y = 1 \Rightarrow -4y^2 + \frac{15}{8}y - 1 = 0$

No real solution

Case 2:

If $\lambda_1, \lambda_2 \neq 0$, then

$$\frac{1}{4} = \frac{x-u}{y-v} = \frac{y}{x} \Rightarrow x = 4y$$

(5) $\Rightarrow 4y^2 = 1 \Rightarrow y = \pm\frac{1}{2}$

$\therefore (x,y) = (2, \frac{1}{2})$ or $(-2, -\frac{1}{2})$

If $(x, y) = (2, \frac{1}{2})$,

$$\frac{2-u}{\frac{1}{2}-v} = \frac{1}{4} \Rightarrow 8 - 4u = \frac{1}{2} - v$$

$$\Rightarrow \begin{cases} -4u + v & = & -\frac{15}{2} \\ u + 4v & = & \frac{15}{8} \end{cases}$$

$$\Rightarrow (u, v) = (\frac{15}{8}, 0)$$

If $(x, y) = (-2, \frac{1}{2})$,

By similar calculation, $(u, v) = (-\frac{225}{136}, \frac{15}{17})$ Comparing the two solutions, f attains minimum at $(x, y, u, v) = (2, \frac{1}{2}, \frac{15}{8}, 0)$

Distance between C and $L = \sqrt{f(2, \frac{1}{2}, \frac{15}{8}, 0)} = \frac{\sqrt{17}}{8}$

12.3.1 Where the Lagrange Multipliers Method "Fails"

Example 12.9. Provided that it exists, find the minimum of $f(x, y) = x$ on:

$$g(x, y) = x^3 - y^2 = 0.$$

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It is easy to see that The absolute minimum of f occurs at $(x, y) = (0, 0)$.

But:

$$\nabla f = [1 \quad 0],$$

while:

$$\nabla g = [3x^2 \quad -2y]$$

Hence,

$$\nabla f = \lambda \nabla g$$

has no solutions.

So, a naive (i.e. fail to check all conditions) application of Lagrange Multipliers would "miss" the point $(0, 0)$ where the minimum occurs.

In general, when ∇g is not necessarily nonzero, one has to separately consider the points where ∇g is zero, after solving for the Lagrange multipliers.

Example 12.10. Provided that it exists, find the maximum of $f(x, y, z) = -y$ on:

$$g_1(x, y, z) = x^2 - y^2 - y^3 - z = 0$$
$$g_2(x, y, z) = y^2 + z = 0$$

The function f in fact attains its absolute maximum at $(0, 0, 0)$ (why?).
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But:

$$\nabla f = [0 \quad -1 \quad 0],$$

while:

$$\nabla g_1 = [2x \quad -2y - 3y^2 \quad -1]$$
$$\nabla g_2 = [0 \quad 2y \quad 1]$$

Hence, there are no λ_1, λ_2 such that:

$$\nabla f(0, 0, 0) = \lambda_1 \nabla g_1(0, 0, 0) + \lambda_2 \nabla g_2(0, 0, 0),$$

which is a direct consequence of the linear dependence of the vectors:

$$\nabla g_1(0, 0, 0) = [0 \quad 0 \quad -1]$$

$$\nabla g_2(0, 0, 0) = [0 \quad 0 \quad 1].$$

(Their span is not "large enough" to accomodate $\nabla f(0, 0, 0)$.)

12.4 Implicit Function Theorem

Question When can we ”solve” a constraint?
For example, if $g(x, y) = c$, can we find $y = h(x)$ such that $g(x, h(x)) = c$?

Example 12.11. Consider level set $g(x, y) = x^2 - y^2 = 0$
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Near $(0, 0), y = x$? $y = -x$? or $\pm|x|$?
y is not uniquely determined by x

Example 12.12. $S : x^2 + y^2 + z^2 = 2$ in $\mathbb{R}^3$
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Question: 3 variables, 1 equation $\Rightarrow$ S is 2-dimensional surface?
Solve for $z = h(x, y)$?
$x = k(y, z)$?

We focus locally near $(0,1,1)$

If we can solve for z as a differentiable function $z=z(x,y)$ near $(0,1,1)$,by implicit differentiation on $x^2+y^2+z^2=2$

$$\frac{\partial}{\partial x}: 2x+2z\frac{\partial z}{\partial x}=0$$

$$\frac{\partial}{\partial y}: 2y+2z\frac{\partial z}{\partial y}=0$$

At $(x,y,z)=(0,1,1) \Rightarrow \begin{cases} 0+2\frac{\partial z}{\partial x} &= 0 \\ 2+2\frac{\partial z}{\partial y} &= 0 \end{cases}$

$\Rightarrow [\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}] = [0 \quad -1]$ at $(0,1,1)$

How about x as a differentiable function $x=x(y,z)$ near $(0,1,1)$?

If so, by implicit differentiation,

$$\frac{\partial}{\partial y}: 2x\frac{\partial x}{\partial y}+2y=0$$

$$\frac{\partial}{\partial z}: 2x\frac{\partial x}{\partial y}+2y=0$$

Put $(x,y,z)=(0,1,1) \Rightarrow \begin{cases} 0+2 &= 0 \quad \text{(coefficient of } \frac{\partial x}{\partial y} \text{ is } \frac{\partial g}{\partial x}=0) \\ 0+2 &= 0 \end{cases}$

Contradiction!

$\therefore$ x is not a differentiable function of y,z near $(0,1,1)$

Reason:

For $x^2+y^2+z^2=2$,

If $y,z>1$ a little bit, no solution for x.

If $y,z<1$ a little bit, 2 solution for x.

Let $g(x,y,z)=x^2+y^2+z^2$.

Difference in the two cases:

At $(0,1,1)$,

$$\frac{\partial g}{\partial z}=2z\neq 0$$

$$\frac{\partial g}{\partial x}=2x=0$$

In general, given constraint $F(x,y,z)=c$

If $z = z(x,y)$, then by implicit differentiation,

$$\left.\begin{array}{lrcl} \frac{\partial}{\partial x}: & \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} & = & 0 \\ \frac{\partial}{\partial y}: & \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} & = & 0 \end{array}\right\} \circledast$$

If $F(\vec{a}) = c, \frac{\partial F}{\partial z}(\vec{a}) \neq 0$, then $\circledast$ has solution (No contradiction)

$\therefore z = z(x,y)$ may exist and

$$\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\frac{\partial F}{\partial z}(\vec{a})} \begin{bmatrix} \frac{\partial F}{\partial x}(\vec{a}) & \frac{\partial F}{\partial y}(\vec{a}) \end{bmatrix}$$

Example 12.13. Multiple Constraints

$$C \left\{ \begin{array}{rcll} x^2 + y^2 + z^2 & = & 2 & \text{3 variables} \\ x + z & = & 1 & \text{2 equations} \end{array} \right.$$

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Question: Is C a 1-dimensional curve? $y = y(x)$? $z = z(x)$?

If we can solve for y, z as differentiable functions $y(x), z(x)$

$$\text{Implicit Differentiation} \Rightarrow \left\{ \begin{array}{rcl} 2x + 2y\frac{dy}{dx} + 2z\frac{dz}{dx} & = & 0 \\ 1 + \frac{dz}{dx} & = & 0 \end{array} \right.$$

$$\begin{bmatrix} 2y & 2z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -2x \\ -1 \end{bmatrix}$$

If this linear system has a solution, then $y = y(x), z = z(x)$ may exist.

For instance, if $(x,y,z) = (0,1,1)$,

$$\begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

In general, given $F_1(x,y,z) = c_1$ and $F_2(x,y,z) = c_2$

Suppose $F_i(a,b,c) = c_i, i = 1,2$.

Do there exist differentiable functions $y = y(x), z = z(x)$ near (a,b,c) such that

$$\left\{ \begin{array}{rcl} F_1(x, y(x), z(x)) & = & c_1 \\ F_2(x, y(x), z(x)) & = & c_2 \end{array} \right. ?$$

If so, by implicit differentiation,

$$\begin{cases} \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\frac{dz}{dx} &= 0 \\ \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\frac{dz}{dx} &= 0 \end{cases}$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -\frac{\partial F_1}{\partial x} \\ -\frac{\partial F_2}{\partial x} \end{bmatrix}$$

If $\begin{bmatrix} \frac{\partial F_1}{\partial y}(\vec{a}) & \frac{\partial F_1}{\partial z}(\vec{a}) \\ \frac{\partial F_2}{\partial y}(\vec{a}) & \frac{\partial F_2}{\partial z}(\vec{a}) \end{bmatrix}^{-1}$ exists at $\vec{a} = (a, b, c)$,

$$\text{then } \begin{bmatrix} \frac{dy}{dx}(a) \\ \frac{dz}{dx}(a) \end{bmatrix} = \begin{bmatrix} \frac{\partial F_1}{\partial y}(\vec{a}) & \frac{\partial F_1}{\partial z}(\vec{a}) \\ \frac{\partial F_2}{\partial y}(\vec{a}) & \frac{\partial F_2}{\partial z}(\vec{a}) \end{bmatrix}^{-1} \begin{bmatrix} -\frac{\partial F_1}{\partial x}(\vec{a}) \\ -\frac{\partial F_2}{\partial x}(\vec{a}) \end{bmatrix}$$

Generally,
given $n + k$ variables
k equations

$$\begin{cases} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) &= c_1 \\ &\vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) &= c_k \end{cases}$$

When can $y_1, \cdots, y_k$ be expressed as functions of $x_1, \cdots, x_n$ locally?