MATH 2010 Chapter 11

11.1 Second Derivative Test

Last time: Definiteness of symmetric matrix

Theorem 11.1. *Suppose* $\Omega \subseteq \mathbb{R}^n$ *is open,* $f : \Omega \to \mathbb{R}$ *is* C^2*, and* $a \in \Omega$ *is a critical point (i.e.* $\nabla f(a) = 0$*).*

If $Hf(a)$ *is:*

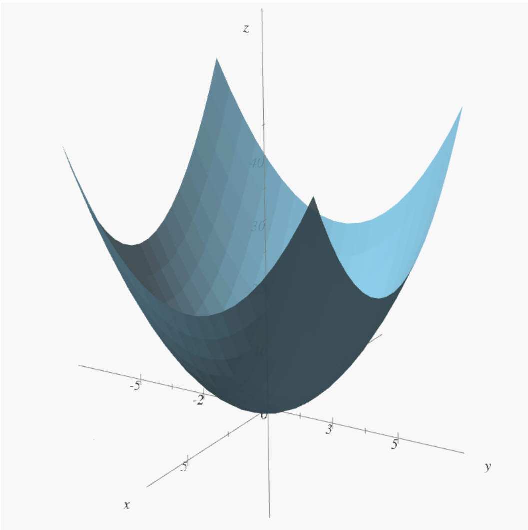
- **positive definite***, then* a *corresponds to a local minimum.*
- **negative definite***, then* a *corresponds to a local maximum.*
- **indefinite***, then* a *is a saddle point.*

Idea of proof:
Use Taylor's Theorem.
$\nabla f(a) = 0 \Rightarrow$ For x near a,

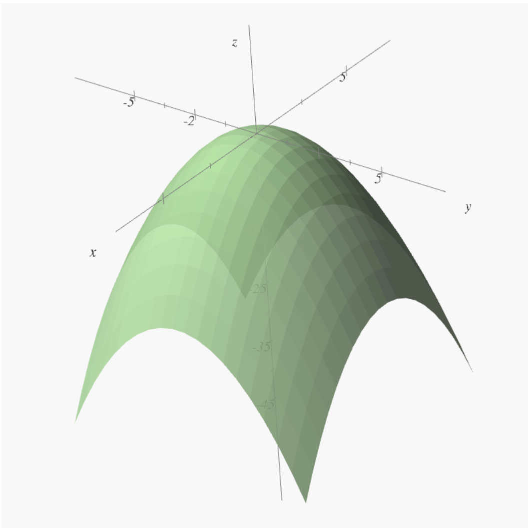
$$f(x) - f(a) \approx \frac{1}{2}(x-a)^T Hf(a)(x-a)$$

If $Hf(a)$ is positive definite
$R.H.S. > 0$ for all $x \neq a$
$\Rightarrow f(x) - f(a) > 0$ for all $x \neq a$ and near a.
$\Rightarrow f$ has a local minimum at a.
"Proof" is similar for the other two cases.
Geometrically,

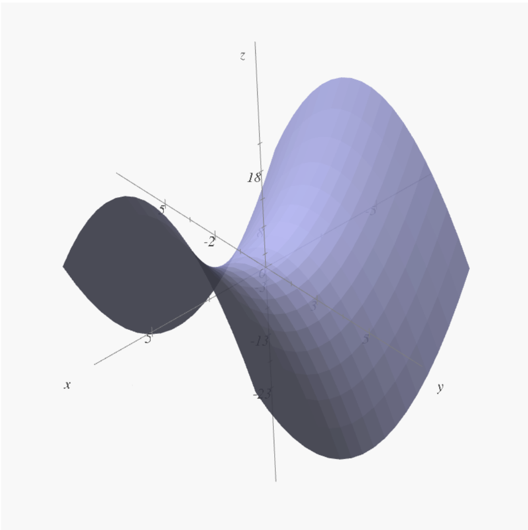
1. $Hf(a)$ is positive definite (e.g. $f = x^2 + y^2$ at $(0,0)$)



2. $Hf(a)$ is negative definite (e.g. $f = -x^2 - y^2$ at $(0,0)$)



3. $Hf(a)$ is indefinite (e.g. $f = x^2 - y^2$ at $(0,0)$)



How do we determine the definiteness of $Hf(a)$?

For the simple case $n = 2$, it can be done easily by completing square.

Theorem 11.2. *Let* $M = \begin{bmatrix} A & B \\ B & C \end{bmatrix}$ *be a symmetric* 2×2 *matrix with real coefficients. Then:*

- M *is positive definite* $\Leftrightarrow \det(M) > 0, A > 0$
- M *is negative definite* $\Leftrightarrow \det(M) > 0, A < 0$
- M *is indefinite* $\Leftrightarrow \det(M) < 0$

Remark. $\det(M) = AC - B^2$.

Proof of Theorem 11.2. Let $q(x, y) = [x \quad y]M \begin{bmatrix} x \\ y \end{bmatrix} = Ax^2 + 2Bxy + Cy^2$

Case I ($A \neq 0$)

$$\begin{aligned} Aq(x, y) &= A^2x^2 + 2ABxy + ACy^2 \\ &= (Ax + By)^2 + (AC - B^2)y^2 \end{aligned}$$

Clearly,

$$\begin{gathered} q(x, y) > 0 \quad \forall (x, y) \neq (0, 0) \Leftrightarrow AC - B^2 > 0, A > 0 \\ q(x, y) < 0 \quad \forall (x, y) \neq (0, 0) \Leftrightarrow AC - B^2 > 0, A < 0 \\ q(x, y) \text{ change signs} \Leftrightarrow AC - B^2 < 0 \end{gathered}$$

Case II ($A = 0$) $AC - B^2 = -B^2 \leq 0$

$$q(x, y) = 2Bxy + Cy^2 = y(2Bx + Cy)$$

Clearly q is neither positive or negative definite and is indefinite $\Leftrightarrow B \neq 0$
$\Leftrightarrow AC - B^2 < 0$ $\square$

Theorem 11.3 (Second Derivative Test for functions of two variables)**.** *If* $\Omega \subseteq \mathbb{R}^2$ *is open,* $f : \Omega \to \mathbb{R}$ *is* C^2*,* $a \in \Omega$*,* $\nabla f(a) = 0$*. Then,*

1. $f_{xx}f_{yy} - f_{xy}^2 > 0, f_{xx} > 0$ *at* $a \Rightarrow a$ *is a local minimum*
2. $f_{xx}f_{yy} - f_{xy}^2 > 0, f_{xx} < 0$ *at* $a \Rightarrow a$ *is a local maximum*
3. $f_{xx}f_{yy} - f_{xy}^2 < 0$ *at* $a \Rightarrow a$ *is a saddle point*
4. $f_{xx}f_{yy} - f_{xy}^2 = 0$ *at* $a \Rightarrow$ *inconclusive*

Remark.

- $f_{xx}f_{yy} - f_{xy}^2 = \det(Hf)$
- In Item 4, the point a can correspond to a local maximum/minimum or saddle point.

Example 11.4.

$$f(x, y) = 3x^2 - 10xy + 3y^2 + 2x + 2y + 3$$

Find and classify critical points of f.

Solution. f is polynomial, so is differentiable on $\mathbb{R}^2$

$$\begin{aligned} \nabla f &= [f_x \quad f_y] \\ &= [6x - 10y + 2 \quad -10x + 6y + 2] \end{aligned}$$

$$\nabla f = \vec{0} \Leftrightarrow \begin{cases} 6x - 10y + 2 = 0 \\ -10x + 6y + 2 = 0 \end{cases}$$
$$\Leftrightarrow (x, y) = (\frac{1}{2}, \frac{1}{2})$$

$\therefore (\frac{1}{2}, \frac{1}{2})$ is the only critical point.

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & -10 \\ -10 & 6 \end{bmatrix}$$

$f_{xx}f_{yy} - f_{xy}^2 = (-6)^2 - (-10)^2 = -64 < 0.$
By 2^{nd} derivative test, $(\frac{1}{2}, \frac{1}{2})$ is a saddle point.

Example 11.5.

$$f(x, y) = 3x - x^3 - 3xy^2$$

Find and classify critical points of f.

Solution. f is a polynomial, so is differentiable on $\mathbb{R}^2$.

$$\begin{aligned}\nabla f &= [f_x \quad f_y] \\ &= [3 - 3x^2 - 3y^2 \quad -6xy]\end{aligned}$$

$$\begin{aligned}&\nabla f = 0 \\ \Leftrightarrow &\begin{cases} 3 - 3x^2 - 3y^2 = 0 \cdots (1) \\ -6xy = 0 \cdots (2) \end{cases}\end{aligned}$$

$(2) \Rightarrow x = 0$ or $y = 0$
If $x = 0$, $(1) \Rightarrow 3 - 3y^2 = 0 \Rightarrow y = \pm 1$
If $y = 0$, $(1) \Rightarrow 3 - 3x^2 = 0 \Rightarrow x = \pm 1$
Hence, there are 4 critical points: $(0, \pm 1), (\pm 1, 0)$.

$$Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -6x & -6y \\ -6y & -6x \end{bmatrix}$$

- $a = (0, 1)$

  Then, $Hf(a) = \begin{bmatrix} 0 & -6 \\ -6 & 0 \end{bmatrix}$.

  $\det Hf(a) = -36 < 0$.

  Hence, the point $(0, 1)$ corresponds to a saddle point.

- $a = (0, -1)$

  Then, $Hf(a) = \begin{bmatrix} 0 & 6 \\ 6 & 0 \end{bmatrix}$.

  $\det Hf(a) = -36 < 0$.

  Hence, the point $(0, -1)$ corresponds to a saddle point.

- $a = (1, 0)$

  Then, $Hf(a) = \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix}$.

  $\det Hf(a) = 36 > 0$.

  $f_{xx}(a) = -6 < 0$.

  Hence $(1, 0)$ corresponds to a local maximum.

- $a = (-1, 0)$

  Then, $Hf(a) = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$.

  $\det Hf(a) = 36 > 0$.

  $f_{xx}(a) = 6 > 0$.

  Hence $(-1, 0)$ corresponds to a local minimum.

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Example 11.6. Inconclusive from 2^{nd} derivative test

$$f(x,y) = x^2 + y^4 \qquad g(x,y) = x^2 - y^4 \qquad h(x,y) = -x^2 - y^4$$
$$\nabla f = [2x \quad 4y^3] \qquad \nabla g = [2x \quad -4y^3] \qquad \nabla h = [-2x \quad -4y^3]$$

$\Rightarrow (0,0)$ is a critical point of f, g, h.

$$Hf = \begin{bmatrix} 2 & 0 \\ 0 & 12y^2 \end{bmatrix} \quad Hg = \begin{bmatrix} 2 & 0 \\ 0 & -12y^2 \end{bmatrix} \quad Hh = \begin{bmatrix} -2 & 0 \\ 0 & -12y^2 \end{bmatrix}$$

$$Hf(0,0) = Hg(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad Hh(0,0) = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Each Hessian matrix has zero determinant at $(0,0)$, so the 2^{nd} derivative test is inconclusive.

Remark. Clearly, f, g, h has local minimum, saddle point and local maximum at $(0,0)$ respectively.

11.1.1 Second Derivative Test for Functions of n Variables

Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ be C^2, $a \in \Omega$, $\nabla f(a) = 0$.

$$Hf(a) = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_n} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1} & f_{x_nx_2} & \cdots & f_{x_nx_n} \end{bmatrix}$$

f is $C^2 \Rightarrow Hf(a)$ is symmetric. From linear algebra, there exists an orthogonal $n \times n$ matrix P (i.e. $P^\top P = I_n$) such that:

$$P^T Hf(a) P = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where λ_i are eigenvalues of $Hf(a)$. Hence,

$$Hf(a) \text{ is } \begin{cases} \text{positive definite} & \Leftrightarrow & \text{All } \lambda_i > 0 \\ \text{negative definite} & \Leftrightarrow & \text{All } \lambda_i < 0 \\ \text{indefinite} & \Leftrightarrow & \text{Some } \lambda_i > 0, \text{some } \lambda_j < 0 \end{cases}$$

11.1.2 Another way to check definiteness of $Hf(a)$

Let H_k be the k by k submatrix

$$H_k = \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_k} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_kx_1} & f_{x_kx_2} & \cdots & f_{x_kx_k} \end{bmatrix}$$

1. $Hf(a)$ is positive definite $\Leftrightarrow$ $\det H_k > 0$ for $k = 1, 2, \cdots, n$
2. $Hf(a)$ is negative definite $\Leftrightarrow$ $\det H_k \begin{cases} < 0 \text{ if } k \text{ is odd} \\ > 0 \text{ if } k \text{ is even} \end{cases}$

For $n = 2$,

$$\det H_1 = \det[f_{xx}] = f_{xx}$$
$$\det H_2 = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

Same result as before.

11.2 Lagrange Multiplier

Finding extrema under constraints.

Example 11.7. Find the point on the parabola $x^2 = 4y$ closest to $(1, 2)$.

Find minimum of $f(x, y) = (x - 1)^2 + (y - 2)^2$ under constraint $g(x, y) = x^2 - 4y = 0$.

(Constraint: expressed as a level set $g = 0$)

IFRAME

https://www.math3d.org/Z2YmkbAD

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Theorem 11.8 (Lagrange Multipliers)**.** *Let f, g be C^1 functions on $\Omega \subseteq \mathbb{R}^n$*

$$S = g^{-1}(c) = \{x \in \Omega : g(x) = c\}$$

Suppose

1. *a corresponds to a local extremum of f on S*

2. $\nabla g(a) \neq 0$

Then $\begin{cases} \nabla f(a) = \lambda \nabla g(a) \text{ for some } \lambda \in \mathbb{R} \\ g(a) = c \end{cases}$

Remark. 1. λ is called a **Lagrange Multiplier**.

2. Let $F(x, \lambda) = f(x) - \lambda(g(x) - c)$
   Then $\nabla F(x, \lambda) = (\underbrace{\nabla(f(x) - \lambda g(x))}_{n \text{ components}}, g(x) - c)$

Find critical points point of f under constraint $g = c$

$$\Updownarrow$$

Find critical point of F without constraint

Back to Example 11.7,

Minimize $f(x, y) = (x-1)^2 + (y-2)^2$

Constraint $g(x, y) = x^2 - 4y = 0$

Solution. f, g are C^1 on $\mathbb{R}^2$.

$$\nabla f = [2(x-1) \quad 2(y-2)]$$

$$\nabla g = [2x \quad -4] \neq \vec{0} \text{ on } \mathbb{R}^2$$

Suppose (x, y) is a local extremum of $f(x, y)$ on $g(x, y) = 0$.
Then, by Lagrange multipliers,

$$\begin{cases} \nabla f(x, y) = \lambda \nabla g(x, y) \text{ for some } \lambda \in \mathbb{R} \\ g(x, y) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2(x-1) = 2\lambda x & \cdots (1) \\ 2(y-2) = -4\lambda & \cdots (2) \\ x^2 - 4y = 0 & \cdots (3) \end{cases}$$

$(1) \Rightarrow x - 1 = \lambda x \Rightarrow x(1-\lambda) = 1$

$(2) \Rightarrow y - 2 = -2\lambda \Rightarrow y = 2(1-\lambda) = \frac{2}{x}$

$(3) \Rightarrow x^2 - \frac{8}{x} = 0, x^3 - 8 = 0 \Rightarrow x = 2$

$\therefore y = \frac{2}{2} = 1$, and now it is easy to check $(x, y) = (2, 1)$ is a solution.

Geometrically, f must have a minimum on $g = 0$.

By the Lagrange Multipliers Theorem, only one point can be that minimum point.

$\Rightarrow f$ has minimum at $(2, 1)$ on $g = 0$.

To summarize, to find the minimum of $f(x, y) = (x-1)^2 + (y-2)^2$ under the constraint $g(x, y) = x^2 - 4y$, we solve:

$$\begin{cases} \nabla f &= \lambda \nabla g \\ g &= 0 \end{cases}$$

Exercise 11.9. Find the point on the parabola $x^2 = 4y$ closest to $(2, 5)$.

$f(x, y) = (x-2)^2 + (y-5)^2$

$g(x, y) = x^2 - 4y$

Remark. The system:

$$\begin{cases} \nabla f &= \lambda \nabla g \\ g &= 0 \end{cases}$$

has solutions. Global minimum on $g = 0 : (4, 4)$.

Not local extremum on $g = 0 : (-2, 1)$.

Example 11.10. Maximize xy^2 on the ellipse

$$x^2 + 4y^2 = 4$$

Solution.

$$\text{Let } f(x, y) = xy^2$$
$$g(x, y) = x^2 + 4y^2$$

Note f is continuous and the ellipse $g = 4$ is closed and bounded.

By EVT, f has global maximum and minimum on $g = 4$.

$$\nabla f = [y^2 \quad 2xy]$$

$$\nabla g = [2x \quad 8y]$$

Note: $\nabla g \neq 0$ on $x^2 + 4y^2 = 4$.

Lagrange multipliers:

$$\begin{cases} \nabla f &= \lambda \nabla g \\ g &= 4 \end{cases} \Leftrightarrow \begin{cases} y^2 &= 2\lambda x & \cdots (1) \\ 2xy &= 8\lambda y & \cdots (2) \\ x^2 + 4y^2 &= 4 & \cdots (3) \end{cases}$$

Case 1: If $y = 0$, then

$$(3) \Rightarrow x^2 = 4 \Rightarrow x = \pm 2,$$
$$\lambda = 0 \text{ by } (1)$$

$\therefore (x, y) = (\pm 2, 0)$.

Case 2: If $y \neq 0$, then:

$$\frac{(2)}{(1)} \Rightarrow \frac{2xy}{y^2} = \frac{8\lambda y}{2\lambda x} \Rightarrow \frac{2x}{y} = \frac{4y}{x} \Rightarrow x^2 = 2y^2$$

By (3), $6y^2 = 4 \Rightarrow y = \pm\sqrt{\frac{2}{3}}$

$\therefore x^2 = 2y^2 = \frac{4}{3} \Rightarrow x = \pm\sqrt{\frac{4}{3}}$

$\therefore (x, y) = (\pm\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}})$.

Compare values of f at the 6 points found using Lagrange Multipliers:

$f(x, y) = xy^2$

$f(\pm 2, 0) = 0$

$f(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}) = \sqrt{\frac{4}{3}} \cdot \frac{2}{3} = \frac{4}{3\sqrt{3}}$ (maximum)

$f(-\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}}) = -\sqrt{\frac{4}{3}} \cdot \frac{2}{3} = -\frac{4}{3\sqrt{3}}$ (minimum)

Hence, for $f(x, y)$ on $g = 4$,

Global maximum value $= \frac{4}{3\sqrt{3}}$ at $(\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}})$

Global minimum value $= -\frac{4}{3\sqrt{3}}$ at $(-\sqrt{\frac{4}{3}}, \pm\sqrt{\frac{2}{3}})$

Remark. We may use another form of Lagrange Multiplier.

Let $F(x, y, \lambda) = f(x, y) - \lambda(g(x, y) - 4) = xy^2 - \lambda(x^2 + 4y^2 - 4)$.

Then, $\nabla F = (y^2 - 2\lambda x, 2xy - 8\lambda y, x^2 + 4y^2 - 4)$, $\nabla F = 0 \Leftrightarrow \begin{cases} y^2 - 2\lambda x &= 0 \\ 2xy - 8\lambda y &= 0 \\ x^2 + 4y^2 - 4 &= 0 \end{cases}$

Same system as before.

For problems of finding maximum/minimum of $f : A \to \mathbb{R}$,

Lagrange Multipliers can be used to study f on ∂A. Consider a previous example:

Example 11.11. Find global maximum/minimum of:

$$f(x,y) = x^2 + 2y^2 - x + 3 \text{ for } x^2 + y^2 \leq 1$$

Solution. Domain $= A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

As found before, f has only one critical point $(\frac{1}{2}, 0)$ in $\text{Int}(A)$, with $f(\frac{1}{2}, 0) = \frac{11}{4}$.

To study f on $\partial A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ by Lagrange Multipliers:

Let $g(x,y) = x^2 + y^2$

$$\nabla g = (2x, 2y) \neq \vec{0} \text{ on } \partial A (g = 1)$$

$$\left\{ \begin{array}{rcl} \nabla f & = & \lambda \nabla g \\ g & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcll} 2x - 1 & = & 2\lambda x & \cdots (1) \\ 4y & = & 2\lambda y & \cdots (2) \\ x^2 + y^2 & = & 1 & \cdots (3) \end{array} \right.$$

$$\begin{aligned} (2) &\Rightarrow (4 - 2\lambda) y = 0 \\ &\Rightarrow \lambda = 2 \text{ or } y = 0 \end{aligned}$$

For $\lambda = 2$:

By (1),

$$\begin{aligned} 2x - 1 &= 4x \\ x &= -\frac{1}{2} \end{aligned}$$

By (3),

$$y = \pm \frac{\sqrt{3}}{2}$$

For $y = 0$,

By (3),

$$x = \pm 1$$

Comparing values of f at five points:

$$f(\frac{1}{2}, 0) = \frac{11}{4}$$

$$f(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = f(-\frac{1}{2}, -\frac{\sqrt{3}}{2}) = \frac{21}{4}$$

$$f(1, 0) = 3$$

$$f(-1, 0) = 5$$

Hence, maximum value $= \frac{21}{4}$ at $(-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$ and minimum value = $\frac{11}{4}$ at $(\frac{1}{2}, 0)$.