MATH 2010 Chapter 1

In one-variable calculus course, you study functions $f(x)$ with both the “input" variable x and “output" value $f(x)$ are real numbers. In this course, we will look at more general functions where the input or output may consist of a tuple of numbers. For example, the function

$$f(x, y, z) = (xy - \cos z, x^2 - y + z)$$

maps the tuple $(2, 1, 0)$ to the tuple $f(2, 1, 0) = (1, 3)$. Tuples like this are called vectors. Here x, y, z are the variables. We say that f is vector-valued multi-variable function.

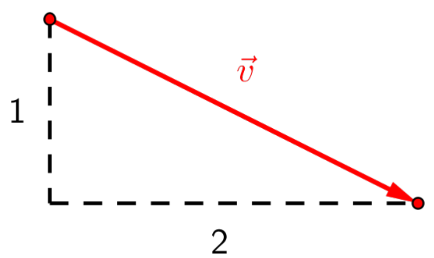
In this chapter, we will discuss vectors and some of its basic properties.

1.1 Euclidean Space

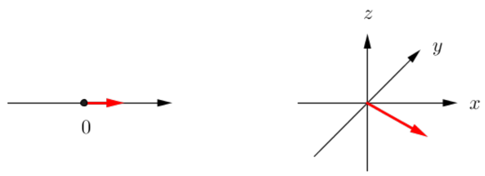
Let $\mathbb{R}$ be the set of real numbers and n be a positive integer. Consider the set

$$\begin{aligned}\mathbb{R}^n &= \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} \ (n \text{ copies of } \mathbb{R}) \\ &= \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}\end{aligned}$$

The set $\mathbb{R}^n$ is called the n-dimensional Euclidean space. Its elements are called n-dimensional vectors or simply vectors. A vector is often written in bold ($\mathbf{v}$), or with an arrow on top ($\vec{v}$). It can be geometrically represented by an arrow. For example, the vector $\vec{v} = (2, -1) \in \mathbb{R}^2$ can be denoted by an arrow that goes to the right by 2 units and goes up by -1 unit, i.e., down by 1 unit, on the plane.



Below are two vectors in $\mathbb{R}$ and $\mathbb{R}^3$. It is more difficult to visualize n-dimensional vectors when $n \geq 4$.



(2) in 1D (1, 3, −4) in 3D

$$n \in \mathbb{N}, \quad \mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \, x_i \in \mathbb{R} \right\}$$

- Each $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ may be viewed as a point or a **vector** in $\mathbb{R}^n$.
- A vector in $\mathbb{R}^n$ is typically denoted by a symbol of the form $\vec{v}$.
- If A and B are points in $\mathbb{R}^n$, then the vector with initial point A and terminal point B is often written as $\overrightarrow{AB}$.
- The vector whose entries are all zero is called the **zero vector**. We denote it by $\vec{0}$.

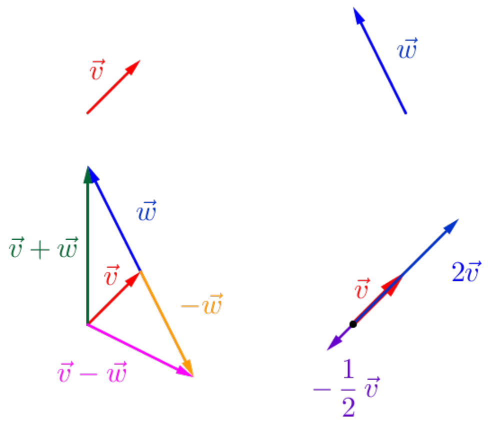
1.2 Basic operations of vectors

Let $\vec{v} = (v_1, v_2, \ldots, v_n), \vec{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ and $r \in \mathbb{R}$. Define

- **Addition Law** $\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2, \cdots, v_n + w_n)$
- **Scalar Multiplication** $r\vec{v} = (rv_1, rv_2 \cdots, rv_n)$
- **Subtraction** $\vec{v} - \vec{w} = v + (-1)\vec{w} = (v_1 - w_1, v_2 - w_2, \cdots, v_n - w_n)$

1.2.1 Geometric Interpretation of Vector Algebra

The algebraic operations defined on vectors can be represented graphically:



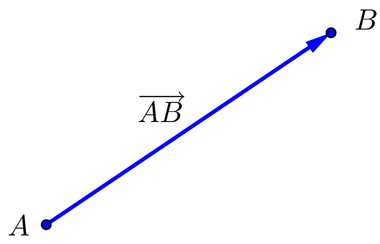
IFRAME

Similar to numbers, there is also a zero vector $\vec{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$ in each dimension n. The zero vectors and the basic operations above have many properties similar to those of numbers.

Proposition 1.1. *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, $\alpha, \beta \in \mathbb{R}$.*

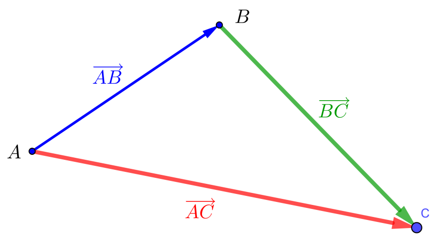
1. $0\vec{v} = \vec{0}$
2. $1\vec{v} = \vec{v}$
3. **Associativity** $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. **Commutativity** $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
5. $\vec{v} + \vec{0} = \vec{v}$
6. **Distributivity** $(\alpha + \beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
7. **Distributivity** $\alpha(\vec{v} + \vec{w}) = \alpha\vec{v} + \alpha\vec{w}$
8. $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

Given two points A and B in $\mathbb{R}^n$. An arrow can be drawn from A to B. It represents a vector which is denoted by $\overrightarrow{AB}$. The point A is called the **initial point** or the **tail** while B is called the **end point** or the **head**.



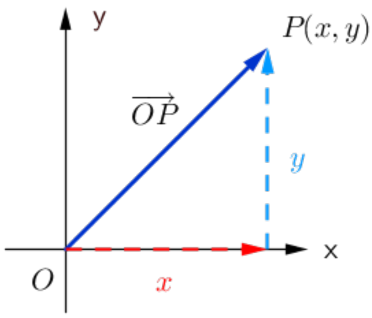
It is clear from the definitions and also the geometric properties that

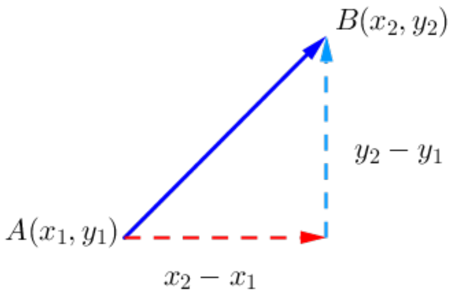
- $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\overrightarrow{BA} = -\overrightarrow{AB}$



A **position vector** is a vector with initial point at the origin. If P has coordinates $(x_1, x_2, \ldots, x_n)$, the position vector is also given by $\overrightarrow{OP} = (x_1, x_2, \ldots, x_n)$.

More generally, the initial point of a vector may not be the origin. For example, consider the vector from $A = (x_1, y_1)$ to $B = (x_2, y_2)$. To move from the initial point to the terminal point, the vector goes to the right by $x_2 - x_1$ and up by $y_2 - y_1$. Hence, $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$.





More generally, the vector from $A = (a_1, a_2, \dots, a_n)$ to $B = (b_1, b_2, \dots, b_n)$ in $\mathbb{R}^n$ is

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n).$$

This formula can also be obtained by considering $\overrightarrow{AB}$ as a difference of position vectors:

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(a_1, a_2, \dots, a_n) + (b_1, b_2, \dots, b_n) \\ &= (b_1 - a_1, b_2 - a_2, \dots, b_n - a_n). \end{aligned}$$

Remark. Besides vectors, an element $x \in \mathbb{R}^n$ can be viewed as a point in the Euclidean space. If we want to describe a location, it is more convenient to think about x as a point. If we want to describe a quantity with both length and direction (e.g. the displacement from one point to another), it is better to think about x as a vector. Some people use notations like $\langle x, y, z \rangle$ for vectors and (x, y, z) for points. We will not follow this convention and write (x, y, z) for both vectors and points.

Example 1.2. Let $A = (1, 0), B = (3, 3), C = (2, 4), D = (0, 1)$ be points on the plane. Show that $ABCD$ is a parallelogram.

Solution.

$$\overrightarrow{AB} = (3,3) - (1,0) = (2,3)$$
$$\overrightarrow{DC} = (2,4) - (0,1) = (2,3) = \overrightarrow{AB}$$

Hence, $ABCD$ is a parallelogram.

Remark. $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are considered equal because they have the same magnitude and direction even though their initial points are different.

1.3 Length and Dot Product

Definition 1.3. The **norm** (or **length**, or **magnitude**) of a vector $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \in \mathbb{R}^n$ is:

$$\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2} = \sqrt{v_1^2 + v_2^2 + \cdots v_n^2}.$$

Definition 1.4. The **dot product** of two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ is:

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} = \sum_{i=1}^{n} v_i w_i.$$

Proposition 1.5. *Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $r \in \mathbb{R}$. Then:*

1. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$ *and* $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.

2. $(r\vec{v}) \cdot \vec{w} = \vec{v} \cdot (r\vec{w}) = r(\vec{v} \cdot \vec{w})$

3. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$

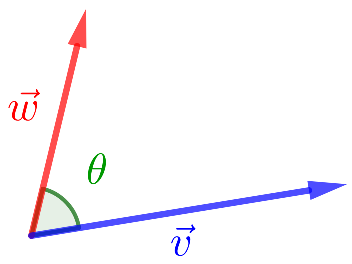
4. $\|r\vec{v}\| = |r|\|\vec{v}\|$

5. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

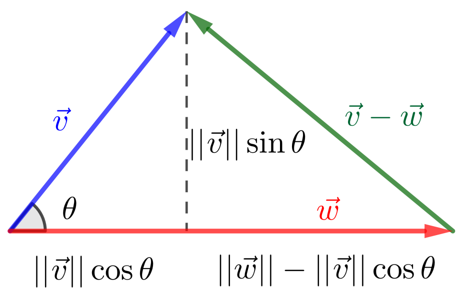
6. $\vec{v} \cdot \vec{v} \geqslant 0$ *with equality* $\vec{v} \cdot \vec{v} = 0$ *occurs if and only if* $\vec{v} = \overrightarrow{0}$.

7. $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$ *where* θ *is the angle between* $\vec{v}$ *and* $\vec{w}$. *Hence, if* $\vec{v}, \vec{w} \neq \vec{0}$, *then:*

$$\vec{v} \cdot \vec{w} = 0 \iff \cos\theta = 0 \iff \vec{v} \perp \vec{w}$$



Proof of Proposition 1.5. We will prove property 7 for the case $n \leq 3$. The proof is essentially the same as that of cosine law. Assume $\theta < \dfrac{\pi}{2}$. Consider the following triangle.



Note:

$$\begin{aligned} ||\vec{v} - \vec{w}||^2 &= (||\vec{v}|| \sin\theta)^2 + (||\vec{w}|| - ||\vec{v}|| \cos\theta)^2 \\ &= ||\vec{v}||^2 \sin^2\theta + ||\vec{w}||^2 - 2||\vec{w}||\,||\vec{v}|| \cos\theta + ||\vec{v}||^2 \cos^2\theta \\ &= ||\vec{v}||^2 + ||\vec{w}||^2 - 2||\vec{w}||\,||\vec{v}|| \cos\theta \qquad (1) \end{aligned}$$

Also,

$$\begin{aligned} ||\vec{v} - \vec{w}||^2 &= (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) \\ &= \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ &= ||\vec{v}||^2 + ||\vec{w}||^2 - 2\vec{v} \cdot \vec{w} \qquad (2) \end{aligned}$$

Compare (1) and (2), we have

$$\vec{v} \cdot \vec{w} = ||\vec{v}||\,||\vec{w}|| \cos\theta.$$

The proof for the cases $\theta \geq \dfrac{\pi}{2}$ can be done similarly. □

Remark. Properties 5 and 7 are geometric properties of length and angle in $\mathbb{R}^n$ for $n \leq 3$. They are used for defining length and angle in higher dimension $n \geq 4$.

A vector of length 1 is called a **unit vector**.

For $\vec{v} \neq \vec{0}$, the vector $\frac{1}{\|\vec{v}\|}\vec{v}$ has length:

$$\left| \frac{1}{\|\vec{v}\|}\vec{v} \right| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1.$$

We call $\frac{1}{\|\vec{v}\|}\vec{v}$ the **unit vector associated with** $\vec{v}$. It captures the direction of $\vec{v}$.

Every nonzero vector $\vec{v}$ has the form:

$$\vec{v} = \lambda\vec{u}, \quad \lambda > 0,$$

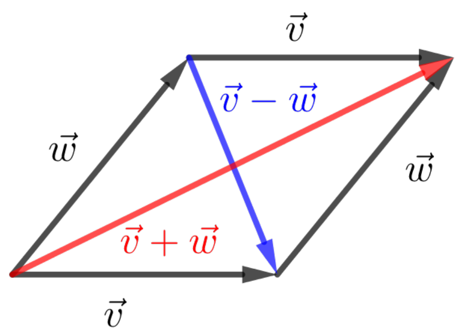
where $\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ is the unit vector associated with $\vec{v}$, and $\lambda = \|\vec{v}\|$ is the length of $\vec{v}$.

Example 1.6. Let $\vec{v}$, $\vec{w}$ have the same length. Show that $(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) = 0$.

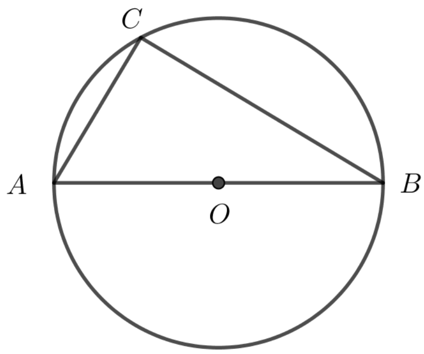
Solution.

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot (\vec{v} - \vec{w}) &= \vec{v} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} - \vec{w} \cdot \vec{w} \\ &= \|v\|^2 - \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{w} - \|w\|^2 \\ &= \|v\|^2 - \|v\|^2 \\ &= 0\end{aligned}$$

Remark. The assumption $||\vec{v}|| = ||\vec{w}||$ means that the parallelogram spanned by $\vec{v}$ and $\vec{w}$ is a rhombus. The computation above shows the fact that the diagonals of a rhombus are perpendicular.



Example 1.7. Consider a circle centered at O. AB is diameter. Show that $\angle ACB$ is a right angle.



Solution.

$$\begin{aligned}
\overrightarrow{AC} &= \overrightarrow{AO} + \overrightarrow{OC} \\
\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC} = -\overrightarrow{AO} + \overrightarrow{OC} \\
\overrightarrow{AC} \cdot \overrightarrow{BC} &= (\overrightarrow{AO} + \overrightarrow{OC}) \cdot (-\overrightarrow{AO} + \overrightarrow{OC}) \\
&= -\overrightarrow{AO} \cdot \overrightarrow{AO} + \overrightarrow{AO} \cdot \overrightarrow{OC} - \overrightarrow{OC} \cdot \overrightarrow{AO} + \overrightarrow{OC} \cdot \overrightarrow{OC} \\
&= -||\overrightarrow{AO}||^2 + ||\overrightarrow{OC}||^2 \qquad (||\overrightarrow{AO}|| = ||\overrightarrow{OC}|| \text{ are radius}) \\
&= 0
\end{aligned}$$

Therefore, $\overrightarrow{AC} \perp \overrightarrow{BC}$. Hence, $\angle ACB$ is a right angle.

Theorem 1.8 (Cauchy-Schwarz Inequality)**.** *For all $\vec{a}, \vec{b} \in \mathbb{R}^n$, the following inequality holds:*

$$\|\vec{a} \cdot \vec{b}\| \leq \|\vec{a}\|\|\vec{b}\|.$$

Remark. For lower dimensional spaces like $\mathbb{R}^2$ and $\mathbb{R}^3$, the inequality follows from the Law of Cosine, since the cosine function has absolute value at most 1.

For $n > 3$, it's not as easy to visualize the situation. We prove the inequaltiy as follows:

Proof of Cauchy-Schwarz Inequality. Observe that for all $t \in \mathbb{R}$, we have:

$$0 \leq \|\vec{a} - t\vec{b}\|^2 = (\vec{a} - t\vec{b}) \cdot (\vec{a} - t\vec{b}) = \|\vec{a}\|^2 - 2(\vec{a} \cdot \vec{b})t + t^2\|\vec{b}\|^2$$

In other words, $\|\vec{a}\|^2$, $-2\vec{a}\cdot\vec{b}$ and $\|\vec{b}\|^2$ are coefficients of a quadratic function which is always non-negative.

The discriminant of such a quadratic function must be non-positive. Hence:

$$(-2(\vec{a}\cdot\vec{b}))^2 - 4\|\vec{a}\|^2\|\vec{b}\|^2 \leq 0$$

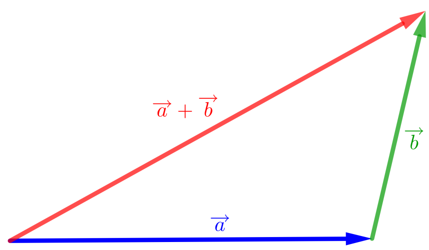
which implies that:

$$\|\vec{a}\cdot\vec{b}\| \leq \|\vec{a}\|\|\vec{b}\|$$

□

Theorem 1.9 (Triangle Inequality)**.** *For any $\vec{a},\vec{b}\in\mathbb{R}^n$, we have:*

$$\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|.$$



Proof of Triangle Inequality.

$$\|\vec{a}+\vec{b}\|^2 = \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b} = \|\vec{a}\|^2 + 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2.$$

By the Cauchy-Schwarz inequality

$$|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\|\vec{b}\|,$$

thus

$$\|\vec{a}+\vec{b}\|^2 \leq \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2 = (\|\vec{a}\| + \|\vec{b}\|)^2.$$

The result follows by taking square roots on both sides. □

1.4 Cross Product

Besides dot product, there is another type of product, called cross product, for vectors in $\mathbb{R}^3$. It can be defined using determinant. Recall the following formulas for 2×2 and 3×3 determinants.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Example 1.10.

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= (1) \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - (2) \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (3) \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (1)[(5)(9) - (6)(8)] - (2)[(4)(9) - (6)(7)] + (3)[(4)(8) - (5)(7)] \\ &= -3 + 12 - 9 \\ &= 0 \end{aligned}$$

Definition 1.11 (Cross product)**.** Let $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$. Their cross product is defined to be

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{\imath} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k} \\ &= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1). \end{aligned}$$

Here the vectors $\hat{\imath} = (1, 0, 0), \hat{\jmath} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$ are the standard unit vectors. A hat instead of an arrow is written on top of each of them to mean that they are unit vectors (vectors of length one).

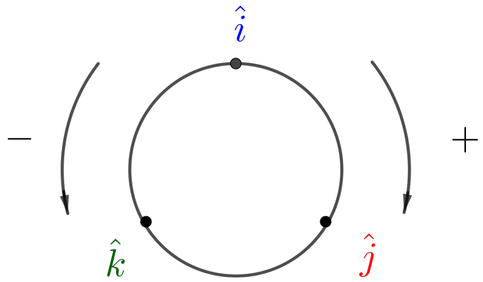
Example 1.12.

$$\begin{aligned} \hat{\imath} \times \hat{\jmath} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{\imath} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \hat{\jmath} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k} \\ &= 0\hat{\imath} - 0\hat{\jmath} + 1\hat{k} = \hat{k} \end{aligned}$$

Similarly, we can compute the cross products of other standard unit vectors:

$$\begin{array}{lll} \hat{\imath} \times \hat{\imath} = \vec{0} & \hat{\imath} \times \hat{\jmath} = \hat{k} & \hat{\imath} \times \hat{k} = -\hat{\jmath} \\ \hat{\jmath} \times \hat{\imath} = -\hat{k} & \hat{\jmath} \times \hat{\jmath} = \vec{0} & \hat{\jmath} \times \hat{k} = \hat{\imath} \\ \hat{k} \times \hat{\imath} = \hat{\jmath} & \hat{k} \times \hat{\jmath} = -\hat{\imath} & \hat{k} \times \hat{k} = \vec{0} \end{array}$$

The diagram below helps you to remember the cross products of standard unit vectors.



Example 1.13. Let $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$. Then

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k} \\ &= -\hat{i} - \hat{j} + \hat{k}\end{aligned}$$

Find $\vec{b} \times \vec{a}$ and $\vec{b} \times \vec{b}$.

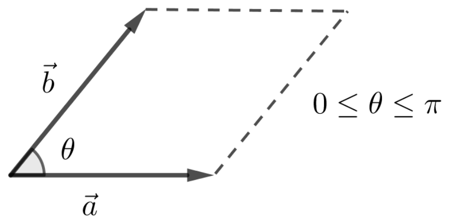
Cross product has the following properties.

Proposition 1.14. *Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$. Then*

1. $\vec{a} \times \vec{a} = \vec{0}$
2. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
3. $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha\vec{a} \times \vec{c} + \beta\vec{b} \times \vec{c}$
4. *Let θ be the angle between $\vec{a}$, $\vec{b}$.*

   $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta =$ *Area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.*

5. $(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$

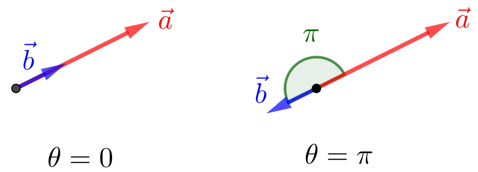


We will prove property 4 below. The other properties can be proved by straightforward computations or properties of determinant.

Remark. From property 4 above,

$$\begin{aligned} \vec{a} \times \vec{b} = \vec{0} \quad &\Leftrightarrow \quad \text{Area of parallelogram} = 0 \\ &\Leftrightarrow \quad \vec{a}, \vec{b} \text{ lie on the same line} \\ &\Leftrightarrow \quad \{\vec{a}, \vec{b}\} \text{ is linearly dependent.} \end{aligned}$$

Hence, two non-zero vectors have zero cross product if and only if they are pointing the same or opposite directions.



Moreover:

- Area of the triangle spanned by $\vec{a}$ and $\vec{b} = \frac{1}{2}\|\vec{a} \times \vec{b}\|$.
- If $\vec{c}, \vec{d} \in \mathbb{R}^2$, then

$$\text{Area of the parallelogram spanned by } \vec{c} \text{ and } \vec{d} = \left| \det \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \right|$$

Proof of Proposition 1.14. By direct expansion,

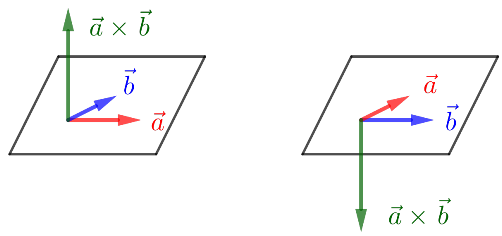
$$\begin{aligned} \|\vec{a} \times \vec{b}\|^2 &= \|\vec{a}\|^2\|\vec{b}\|^2 - (\vec{a} \cdot \vec{b})^2 \\ &= \|\vec{a}\|^2\|\vec{b}\|^2(1 - \cos^2\theta) \\ &= \|\vec{a}\|^2\|\vec{b}\|^2 \sin^2\theta \end{aligned}$$

Since $0 \le \theta \le \pi$, we have $\sin\theta \ge 0$ and so

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \, \|\vec{b}\| |\sin\theta| = \|\vec{a}\| \, \|\vec{b}\| \sin\theta$$

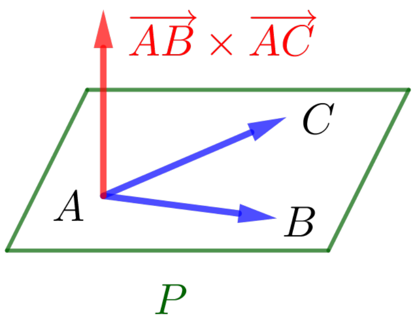
□

Suppose $\vec{a} \times \vec{b}$ are non-zero. Then $\vec{a}$ and $\vec{b}$ are both non-zero. From property 5 above, $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. It can be shown that $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy the **right hand rule**.



Example 1.15. Let $A = (1, 2, 1)$, $B = (1, -1, 0)$ and $C = (2, 3, 2)$ be points on a plane P. Find a normal vector of P, i.e. a vector perpendicular to P.

Solution. The line segments AB and AC both lie on P. Hence, the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ is perpendicular to P.



$$
\begin{aligned}
\overrightarrow{AB} &= (1,-1,0)-(1,2,1)=(0,-3,-1)\\
\overrightarrow{AC} &= (2,3,2)-(1,2,1)=(1,1,1)\\
\overrightarrow{AB}\times\overrightarrow{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ 0 & -3 & -1\\ 1 & 1 & 1 \end{vmatrix}\\
&= \begin{vmatrix} -3 & -1\\ 1 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 0 & -1\\ 1 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 0 & -3\\ 1 & 1 \end{vmatrix}\hat{k}\\
&= [(-3)(1)-(-1)(1)]\,\hat{i} - [(0)(1)-(-1)(1)]\,\hat{j} + [(0)(1)-(-3)(1)]\,\hat{k}\\
&= -2\hat{i}-\hat{j}+3\hat{k}
\end{aligned}
$$

Therefore, $(-2,-1,3)\perp P$.

Another product closely related to cross product is also defined for vectors in $\mathbb{R}^3$.

Definition 1.16. The **triple product** of $\vec{a},\vec{b},\vec{c}\in\mathbb{R}^3$ is defined to be $\vec{a}\cdot(\vec{b}\times\vec{c})$.

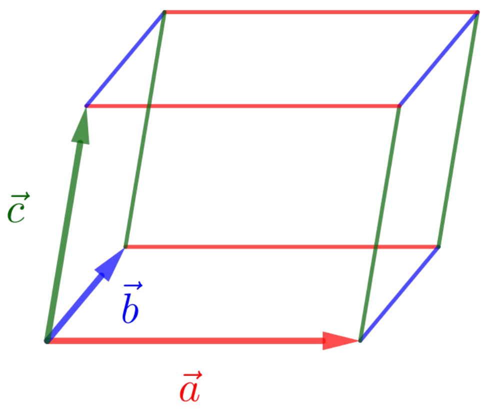
From this definition, it is easy to see that

$$
\begin{aligned}
\vec{a}\cdot(\vec{b}\times\vec{c}) &= (a_1,a_2,a_3)\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3 \end{vmatrix}\\
&= (a_1,a_2,a_3)\cdot\left(\begin{vmatrix} b_2 & b_3\\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} b_1 & b_3\\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & b_2\\ c_1 & c_2 \end{vmatrix}\right)\\
&= a_1\begin{vmatrix} b_2 & b_3\\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3\\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2\\ c_1 & c_2 \end{vmatrix}\\
&= \begin{vmatrix} a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3 \end{vmatrix}
\end{aligned}
$$

It follows from this formula that a triple product depends on the order of its factors. From properties of determinant,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -\vec{b}\cdot(\vec{a}\times\vec{c}) = -\vec{c}\cdot(\vec{b}\times\vec{a})$$

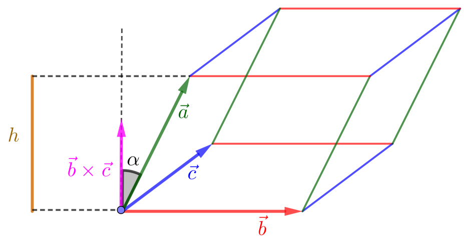
Proposition 1.17. *Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ in $\mathbb{R}^3$ determine a parallelepiped as below.*



Its volume can be computed using triple product:

$$|\vec{a} \cdot (\vec{b} \times \vec{c})| = \textit{Volume of parallelepiped spanned by } \vec{a}, \vec{b}, \vec{c}.$$

Proof of Proposition 1.17. Consider the parallelogram spanned by $\vec{b}$ and $\vec{c}$ as the base of the parallelopiped. Let α be the angle between $\vec{a}$ and $\vec{b} \times c$. Suppose $\alpha \leqslant \pi/2$.



Then:

$$\begin{aligned} \vec{a} \cdot (\vec{b} \times \vec{c}) &= \|\vec{a}\| \|\vec{b} \times \vec{c}\| \cos \alpha \\ &= \|\vec{b} \times \vec{c}\| h \\ &= \text{ Base Area } \times \text{ height} \\ &= \text{ Volume of parallelepiped} \end{aligned}$$

The case for $\pi/2 < \alpha \leqslant \pi$ can be done similarly. In that case,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = - \text{ Volume of parallelepiped}$$

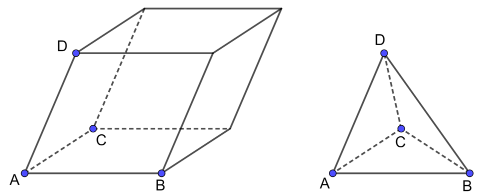
$\square$

Remark.

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_1 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0 \Longleftrightarrow \text{Volume of parallelepiped} = 0$$

$$\Longleftrightarrow \{\vec{a}, \vec{b}, \vec{c}\} \text{ is linearly dependent}$$

Consider a tetrahedron with vertices $A, B, C, D \in \mathbb{R}^3$. To find a formula of its volume, we compare the tetrahedron with the parallelopiped spanned by $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.



$$\begin{aligned}
\text{Volume of Tetrahedron} &= \frac{1}{3} \cdot \text{Area}(\triangle ABC) \cdot \text{height} \\
&= \frac{1}{3} \cdot \frac{1}{2} \cdot (\text{Area of parallelogram spanned by } \overrightarrow{AB}, \overrightarrow{AC}) \cdot \text{height} \\
&= \frac{1}{6} \cdot \text{Volume of Parallelepiped} \\
&= \frac{1}{6} \left| \overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) \right|
\end{aligned}$$

Example 1.18. Let $A = (1,0,1), B = (1,1,2), C = (2,1,1), D = (2,1,3)$. Find the volume of the tetrahedron $ABCD$.

Solution.

$$\begin{aligned}
\overrightarrow{AB} &= (1,1,2) - (1,0,1) = (0,1,1) \\
\overrightarrow{AC} &= (2,1,1) - (1,0,1) = (1,1,0) \\
\overrightarrow{AD} &= (2,1,3) - (1,0,1) = (1,1,2)
\end{aligned}$$

Their triple product is:

$$(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -2$$

and so:

$$\text{Volume of the tetrahedron } ABCD = \frac{1}{6} \cdot |2| = \frac{1}{3}$$