Discrete Fractal Dimensions of the Ranges of Random Walks Associate with Random Conductances

Xinghua Zheng

Department of ISOM, HKUST

http://ihome.ust.hk/∼xhzheng/

International Conference on Advances on Fractals and Related Topics, Dec 2012

Based on Joint Work with Yimin Xiao

Outline

The Random Conductance Model

- $\mathbb{Z}^d$ = d-dimension integer lattice; E_d = {non-oriented nearest neighbor bonds}
- **Environment**: for a given distribution $\mathbb{Q}$ on $[0, \infty)$,

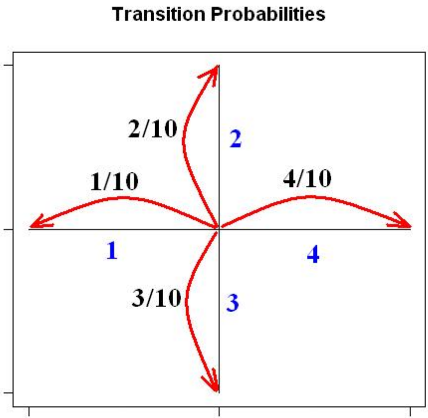
$$\mu_e \sim_{i.i.d.} \mathbb{Q}, \qquad \text{for all } e \in E_d;$$

- Given a realization $\omega = \{\mu_e : e \in E_d\}$, two random walks:
  1. Variable speed random walk (VSRW), (X_t), waits at x for an exponential time with mean $1/\mu_x$;
  2. Constant speed random walk (CSRW), (Y_t), waits at x for an exponential time with mean 1;

  and then jumps to a neighboring site y with probability

$$P_{xy}(\omega) = \frac{\mu_{xy}}{\mu_x} \quad \text{where } \mu_x = \sum_{y \sim x} \mu_{xy}.$$

Transition Probabilities



Examples

Eg 1:

- $\mathbb{Q} = \delta_{\{1\}}$, then μ_e are constantly 1, and Y_t is just the usual nearest neighbor random walk
- Functional CLT (FCLT):

$$\frac{Y_{nt}}{\sqrt{n}} \Rightarrow B_t.$$

Eg 2:

- $\mathbb{Q} =$ Bernoulli(p), then Y_t is a simple random walk on the connected component of percolation

Eg 3:

- $\mathbb{Q}$ supported on $[1, \infty)$ – what we shall focus on

Two laws

- Two laws:
  1. **Quenched Law**: For any given realization ω, study the law P_ω of $(X_t)/(Y_t)$ under this realization
  2. **Averaged (or Annealed) Law**: the law by taking expectation of the quenched law P_ω w.r.t. $\mathbb{P}$
- Focus on quenched law P_ω
- Basic Questions: the long run behavior of $(X_t)/(Y_t)$, e.g.,
  1. does the quenched FCLT (QFCLT) hold?
  2. What about the fractal properties of the sample paths of $(X_t)/(Y_t)$?

QFCLT

- [Barlow and Deuschel(2010)] For the VSRW X, when $d \geq 2$, for $\mathbb{P}$-a.a. ω, under P_0^ω, $X_{n^2 t}/n \Rightarrow \sigma_V B_t$, where σ_V is non-random, and B_t is a standard d-dimensional Brownian-motion.
- [Barlow and Deuschel(2010)] For the CSRW Y, when $d \geq 2$, for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^2 t}/n \Rightarrow \sigma_C B_t$,
$$\text{where } \sigma_C = \begin{cases} \sigma_V/\sqrt{2d\mathbb{E}\mu_e}, & \text{if } \mathbb{E}\mu_e < \infty, \\ 0, & \text{if } \mathbb{E}\mu_e = \infty. \end{cases}$$
- [Barlow and Černý(2011)], [Černý(2011)] For the CSRW Y, when $d \geq 2$ and $\mathbb{Q}(\mu_e \geq u) \sim C/u^\alpha$ for some $\alpha \in (0,1)$, then for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^{2/\alpha} t}/n$ converges to a multiple of the fractional kinetics process;
- [Barlow and Zheng(2010)] For the CSRW Y, when $d \geq 3$ and $\mathbb{Q}$ is Cauchy tailed, then for $\mathbb{P}$-a.a. ω, under P_0^ω, $Y_{n^2 (\log n)\, t}/n$ converges to a multiple of a d-dimensional Brownian-motion.

Discrete Hausdorff Dimension

- For any $n \in \mathbb{N}$, let $V_n = V(0, 2^n)$ be the cube of side length 2^n centered at $0 \in \mathbb{Z}^d$, and $S_n := V_n \setminus V_{n-1}$
- For any set $B \subseteq \mathbb{Z}^d$, let $s(B)$ be its side length
- [Barlow and Taylor(1992)] For any measure function h and any set $A \subseteq \mathbb{Z}^d$, the **discrete Hausdorff measure** of A w.r.t h is

$$m_h(A) = \sum_{n=1}^{\infty} \nu_h(A, S_n).$$

where

$$\nu_h(A, S_n) = \min\left\{ \sum_{i=1}^{k} h\Big(\frac{s(B_i)}{2^n}\Big) : A \cap S_n \subset \bigcup_{i=1}^{k} B_i \right\}.$$

- For $\alpha > 0$, define $h(r) = r^\alpha$, and let $m_\alpha(A) = m_h(A)$. Then the **discrete Hausdorff dimension** of A is given by

$$\dim_{\mathrm{H}} A = \inf\left\{\alpha > 0 : m_\alpha(A) < \infty\right\}.$$

Discrete Packing Dimension

- [Barlow and Taylor(1992)] For any measure function h, $\varepsilon > 0$, and any set $A \subseteq \mathbb{Z}^d$, the **discrete packing measure** of A w.r.t h is

$$p_h(A, \varepsilon) = \sum_{n=1}^{\infty} \tau_h(A, S_n, \varepsilon),$$

where

$$\tau_h(A, S_n, \varepsilon) = \max \left\{ \sum_{i=1}^{k} h\left(\frac{r_i}{2^n}\right) : x_i \in A \cap S_n,\ V(x_i, r_i) \text{ disjoint},\ 1 \le r_i \le 2^{(1-\varepsilon)n} \right\}$$

- Say that $A \subseteq \mathbb{Z}^d$ is *h-packing finite* if $p_h(A, \varepsilon) < \infty$ for all $\varepsilon \in (0, 1)$.
- The **discrete packing dimension** of A is defined by

$$\dim_{\mathrm{P}} A = \inf \left\{ \alpha > 0 : A \text{ is } r^{\alpha}\text{-packing finite} \right\}.$$

Discrete Dimensions of the Range of RCM

Theorem

[Xiao and Zheng(2011)] Let

$$\mathrm{R} = \{x \in \mathbb{Z}^d : X_t = x \text{ for some } t \geq 0\}$$

be the range of VSRW X (as well as that of CSRW Y). Assume that $d \geq 3$ and $\mathbb{Q}(\mu_e \geq 1) = 1$. Then for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$\dim_{\mathrm{H}}\mathrm{R} = \dim_{\mathrm{P}}\mathrm{R} = 2, \quad \mathrm{P}_0^{\omega}\text{-a.s.}.$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$ denote respectively the discrete Hausdorff and packing dimension.

Recurrent/Transient Sets for RCM

Theorem

[Xiao and Zheng(2011)] Assume that $d \geq 3$ and $\mathbb{P}(\mu_e \geq 1) = 1$. Let $A \subset \mathbb{Z}^d$ be any (infinite) set. Then for $\mathbb{P}$-almost every $\omega \in \Omega$, the following statements hold.

(i) *If $\dim_{\mathrm{H}} A < d - 2$, then*

$$\mathrm{P}_0^\omega\big(X_t \in A \ \text{ for arbitrarily large } t > 0\big) = 0.$$

(ii) *If $\dim_{\mathrm{H}} A > d - 2$, then*

$$\mathrm{P}_0^\omega\big(X_t \in A \ \text{ for arbitrarily large } t > 0\big) = 1.$$

Remark

Both theorems are also proven for the Bouchaud's trap model.

Main Ingredients of Proof

- Basic idea: derive various estimates for ordinary random walks used in [Barlow and Taylor(1992)], by using general Markov chain techniques
- Main ingredients:
  1. Gaussian heat kernel bounds for the VSRW ([Barlow and Deuschel(2010)]);
  2. Hitting probability estimates;
  3. Tail probability estimates of the sojourn measure for the discrete time VSRW;
  4. Tail probability estimates of the maximal displacement of VSRW;
  5. A SLLN for dependent events;
  6. A zero-one law as a consequence of an elliptic Harnack inequality that the VSRW satisfies.

Proof Sketch for Theorem 1

- $\dim_{\mathrm{P}} \mathrm{R} \le 2$ P_0^ω-a.s.: first moment argument;
- $\dim_{\mathrm{H}} \mathrm{R} \ge 2$ P_0^ω-a.s.: let $\widehat{\mathrm{R}}$ be the range of the discrete time VSRW $(\widehat{Y}_n) := (Y_n)$, and show that $\dim_{\mathrm{H}} \widehat{\mathrm{R}} \ge 2$.
  - Let μ be the counting measure on $\widehat{\mathrm{R}}$. Show that
    $$\mu\big(Q_k(x)\big) \le c\, n\, 2^{2k} \qquad \text{for every } x \in S_n \text{ and } 0 \le k \le n.$$
  - Frostman's lemma $\Rightarrow$
    $$\nu_2\big(\widehat{\mathrm{R}}, S_n\big) \ge c^{-1}\, n^{-1} 2^{-2n}\, \mu(S_n)$$
  - Hitting probability estimate $\Rightarrow$
    $$\mathrm{E}_0^\omega\big(\mu(S_n)\big) \ge c\, 2^{2n}$$
    and hence $\mathrm{E}_0^\omega\big(m_2(\widehat{\mathrm{R}})\big) = \infty$.
  - To further prove $m_2\big(\widehat{\mathrm{R}}\big) = \infty$ P_0^ω-a.s., let $n_k = \lfloor \lambda k \log k \rfloor$ for $\lambda > 0$ TBD, and define
    $$\tau_k = \inf\Big\{n > 0 : \ \widehat{X}_n \notin V\big(0, 2^{n_k}\big)\Big\}.$$
    Show that
    1. $\mathrm{P}_0^\omega\Big(|\widehat{X}_{\tau_{k-1}}| > 2^{n_k - 3}\Big) \le c \exp(-ck)$; and
    2. On the event $\{|\widehat{X}_{\tau_{k-1}}| \le 2^{n_k-3}\}$,
       $$\mathrm{P}^\omega_{\widehat{X}_{\tau_{k-1}}}\Big(\mu(S_{n_k}) \ge c\, 2^{2n_k}\Big) \ge p.$$
    3. The SLLN for dependent event concludes.

Summary

0. QFCLT for the VSRW/CSRW
1. Discrete fractal dimensions of the range of VSRW/CSRW
2. Characterization of recurrent/transient sets for VSRW/CSRW
3. Similarly for Bouchaud's trap model.

Thank you!

- Barlow, M. T. and Černý, J. (2011), "Convergence to fractional kinetics for random walks associated with unbounded conductances," *Probab. Theory Related Fields*, 149, 639–673.
- Barlow, M. T. and Deuschel, J.-D. (2010), "Invariance principle for the random conductance model with unbounded conductances," *Ann. Probab.*, 38, 234–276.
- Barlow, M. T. and Taylor, S. J. (1992), "Defining fractal subsets of $\mathbf{Z}^d$," *Proc. London Math. Soc. (3)*, 64, 125–152.
- Barlow, M. T. and Zheng, X. (2010), "The random conductance model with Cauchy tails," *Ann. Appl. Probab.*, 20, 869–889.
- Černý, J. (2011), "On two-dimensional random walk among heavy-tailed conductances," *Electron. J. Probab.*, 16, no. 10, 293–313.
- Xiao, Y. and Zheng, X. (2011), "Discrete Fractal Dimensions of the Ranges of Random Walks in $\mathbb{Z}^d$ Associate with Random

Conductances," to appear in *Probability Theory and Related Fields*.