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Modeling Potential as Fiber Entropy and Pressure as Entropy

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topological dynamical system (TDS) (X, T): a cpt metric space X with metric d and a continuous surjection $T : X \to X$

one of central problems: classification

- usually looking for isomorphism invariants, i.e.
    - properties, e.g. ergodicity, mixing, $\cdots$

      isomorphic $\Longrightarrow$ have the property or not simultaneously
    - objects, e.g. numbers, groups, $\cdots$

      isomorphic $\Longrightarrow$ equal numbers, isomorphic groups, $\cdots$

among the most important ones: entropy

history of entropy

- measure-theoretic entropy for measurable dynamical systems: Kolmogorov (then Sinai, $\cdots$)
- topological entropy for TDS: Adler-Konheim-McAndrew (then R. Bowen, Dinaburg, $\cdots$)
- classical variational principle by Goodman and Goodwyn (then Misiurewicz, $\cdots$)

$$h_{\text{top}}(X, T) = \sup\{h_\mu(X, T) : \mu \text{ invariant}\}$$

topological pressure of potentials: useful in statistical mechanics, ergodic theory, dynamical systems, $\cdots$

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- general TDS: Walters

topological pressure of potentials: useful in statistical mechanics, ergodic theory, dynamical systems, $\cdots$

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- natural "generalization" of entropy
  - $P(X, T, c) = h_{\text{top}}(X, T) + c, \forall c \in \mathbb{R}$ (so $P(X, T, 0) = h_{\text{top}}(X, T)$)
  - variational principle for potentials generated by continuous f

$$P(X, T, f) = \sup \left\{ h_\mu(X, T) + \int_X f d\mu : \mu \text{ invariant} \right\},$$

generalizing classical variational principle about entropy our motivation:

ENTROPY $\underset{\text{difference?}}{\overset{\text{what}}{\rule{4cm}{0.4pt}}}$ PRESSURE

a potential arising naturally from TDSs:

assume: (X, T), a relative symbolic extension of (Y, S)

- i.e. $(X, T) \subset (Y, S) \times (Z, \sigma)$ projects onto (Y, S) (with projection π), where (Z, σ) is a symbolic system with standard clopen partition $\mathcal{U}_Z$

introduce fiber entropy potentials (as functions over (Y, S)) by

$$\begin{aligned} H_n(y) &= \log\left(\text{minimal cardinality of } \mathcal{V} \subset \bigvee_{i=0}^{n-1} T^{-i}\mathcal{U} \text{ covering } \pi^{-1}(y)\right) \\ &= \text{number of } n\text{-length words in } \pi^{-1}(y) \end{aligned}$$

for $n \in \mathbb{N}$ and $y \in Y$, where $\mathcal{U} = (Y \times \mathcal{U}_Z) \cap X$

properties of fiber entropy potentials $\mathfrak{H} = \{H_n : n \in \mathbb{N}\}$

- nonnegative, upper semicontinuous (over Y) and nondecreasing (with respect to n)
- subadditive in the sense of

$$H_{n+m}(y) \leq H_n(y) + H_m(S^n y), n, m \in \mathbb{N}, y \in Y$$

- (Downarowicz-Huczek-Z, preprint) topological pressure of potentials $\mathfrak{H}$ is just entropy

$$P(Y, S, \mathfrak{H}) = h_{\text{top}}(X, T) = \sup\left\{h_\nu(Y, S) + \lim_{n\to\infty} \frac{1}{n}\int_Y H_n d\nu : \nu \text{ invariant}\right\}$$

$\Rightarrow$ "direct" proof (without using variational principle concerning topological pressure)

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  - "direct" proof (without using variational principle concerning topological pressure)
    - subadditive ergodic theorem by Kingman + inner variational principle of relative entropy by Downarowicz-Serafin

Theorem (Downarowicz-Huczek-Z, preprint)

each nonnegative, upper semicontinuous, subadditive potential $\mathfrak{F} = \{f_n : n \in \mathbb{N}\}$ (over (Y, S)) is equivalent to a fiber entropy potential $\mathfrak{H}$ (determined by a relative symbolic extension (X, T) of (Y, S)) in the sense that $P(Y, S, \mathfrak{H}) = P(Y, S, \mathfrak{F})$ and

$$\lim_{n\to\infty} \frac{1}{n} \int_Y H_n d\nu = \lim_{n\to\infty} \frac{1}{n} \int_Y f_n d\nu \text{ for invariant } \nu.$$

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applications (combined with previous observation about fiber entropy potentials):

- (applying to $f + ||f||$ for continuous f) variational principle of topological pressure for f by Walters, 75
- (applying its relative version) a relativised variational principle of topological pressure by Ledrappier-Walters, 77
- variational principle of topological pressure for subadditive potentials by Cao-Feng-Huang, 08
    - our results ONLY work for NONNEGATIVE case

for $\mathfrak{F} = \{f_n : n \in \mathbb{N}\}$, a potential over (Y, S) with metric d, set

$$S_{n,\epsilon}(\mathfrak{F}) = \sup \left\{ \sum_{y \in E} 2^{f_n(y)} : E \subset Y \text{ is } (n,\epsilon)\text{-separated} \right\},$$

$$R_{n,\epsilon}(\mathfrak{F}) = \inf \left\{ \sum_{y \in F} 2^{f_n(y)} : F \subset Y \text{ is } (n,\epsilon)\text{-spanning} \right\} \leq S_{n,\epsilon}(\mathfrak{F}),$$

where

- $d_n(x_1, x_2) = \max_{0}^{n-1} d(S^i x_1, S^i x_2)$
- (n, ϵ)-spanning $E \subset Y$ if $\forall x_1 \in Y, \exists x_2 \in E$ with $d_n(x_1, x_2) < \epsilon$
- (n, ϵ)-separated $F \subset Y$ if $d_n(x_1, x_2) \geq \epsilon$ once $x_1 \neq x_2 (\in F)$

introduce

$$P(Y,S,\mathfrak{F}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log S_{n,\epsilon}(\mathfrak{F}),$$

$$\underline{P}(Y,S,\mathfrak{F}) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log S_{n,\epsilon}(\mathfrak{F}) \leq P(Y,S,\mathfrak{F}),$$

$$Q(Y,S,\mathfrak{F}) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log R_{n,\epsilon}(\mathfrak{F}) \leq P(Y,S,\mathfrak{F}),$$

$$\underline{Q}(Y,S,\mathfrak{F}) = \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log R_{n,\epsilon}(\mathfrak{F}) \leq \min\{Q(Y,S,\mathfrak{F}), \underline{P}(Y,S,\mathfrak{F})\}$$

- (Walters, 75) if $\mathfrak{F}$ is generated by continuous f, then, by uniform continuity of f,

$$P(Y,S,\mathfrak{F}) = \underline{P}(Y,S,\mathfrak{F}) = Q(Y,S,\mathfrak{F}) = \underline{Q}(Y,S,\mathfrak{F})$$

byproduct of main results:

- for nonnegative, upper semicontinuous, subadditive potential $\mathfrak{F}$, $P(Y, S, \mathfrak{F}) = \underline{P}(Y, S, \mathfrak{F})$
  - first observe it for fiber entropy potential, then proceed a similar argument of Main Theorem's proof
- it is possible (comparing to Walter's result)

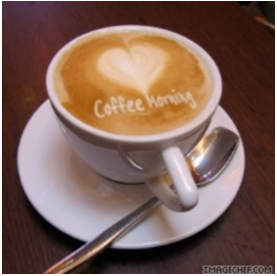
$$P(Y, S, \mathfrak{F}) = \underline{P}(Y, S, \mathfrak{F}) > Q(Y, S, \mathfrak{F}) = \underline{Q}(Y, S, \mathfrak{F}) = 0$$

  - for some nonnegative, upper semicontinuous and additive potential (realized by a fiber entropy potential)
    - Toeplitz system with positive entropy which is relatively independent almost 1-1 extension over the odometer
  - for some nonnegative, continuous, subadditive potential
    - modify the above fiber entropy potential (improve its continuity and destroy its additivity)

Remaining Question:

how about a general upper semicontinuous, subadditive potential (which may be not nonnegative)?

Thank you!



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