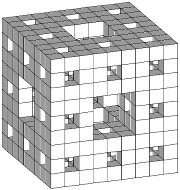
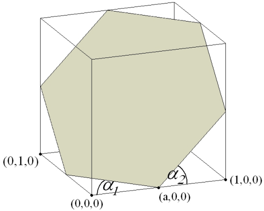
Slices through self-similar sets

Rüdiger Zeller,
joint work with Christoph Bandt,
Greifswald, Germany

December 14, 2012

International Conference on Advances on Fractals & Related Topics



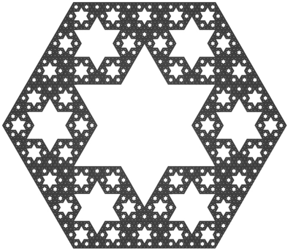


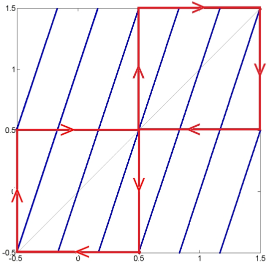
Finite orbits in branching dynamical systems

Relation to slices through self-similar sets

Slices of finite type

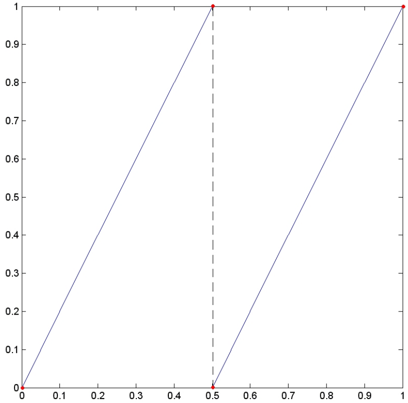
The golden dodecahedron





Finite orbits for multivalued maps

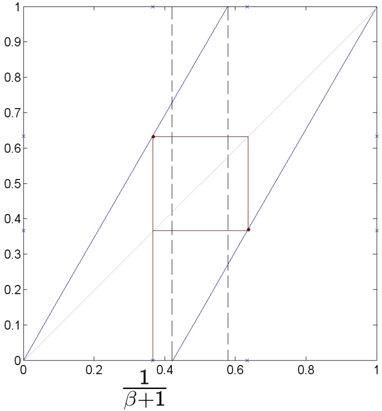
$g(x) = 2x \mod 1$



All orbits of $\frac{1}{2}$ are finite, $g(\frac{1}{2}) = \{0, 1\} = g^n(\frac{1}{2})$.

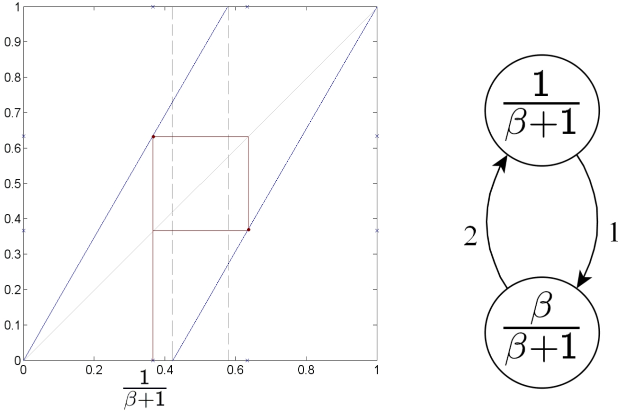
Finite orbits for multivalued maps

$$g_1(x) = \beta x, \quad g_2(x) = \beta x + 1 - \beta, \quad \beta = \sqrt{3}.$$



Finite orbits for multivalued maps

$$g_1(x) = \beta x, \quad g_2(x) = \beta x + 1 - \beta, \quad \beta = \sqrt{3}.$$



The orbit of $\frac{1}{\beta+1}$ is finite, $g_1(\frac{1}{\beta+1}) = \frac{\beta}{\beta+1}$ and $g_2(\frac{\beta}{\beta+1}) = \frac{1}{\beta+1}$.

Definition (BDS)

- A branching dynamical system is given by a set of mappings: $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R},\ j = 1, \ldots, k\}$.

Definition (BDS)

- A branching dynamical system is given by a set of mappings: $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R},\ j = 1, \ldots, k\}$.
- The set of successors of nth generation is given by

$$B^n(x) = B(B^{n-1}(x)), \quad \text{where} \quad B(x) = \{g_j(x) \mid x \in I_j\}.$$

Definition (BDS)

- A branching dynamical system is given by a set of mappings: $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R},\ j = 1, \dots, k\}$.
- The set of successors of nth generation is given by

$$B^n(x) = B(B^{n-1}(x)), \quad \text{where} \quad B(x) = \{g_j(x) \mid x \in I_j\}.$$

- The set of successors produced by x is given by

$$B^\infty(x) = \bigcup_{n=0}^{\infty} B^n(x).$$

Definition (BDS)

- A branching dynamical system is given by a set of mappings: $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R},\, j = 1, \dots, k\}$.
- The set of successors of nth generation is given by

$$B^n(x) = B(B^{n-1}(x)), \quad \text{where} \quad B(x) = \{g_j(x) \mid x \in I_j\}.$$

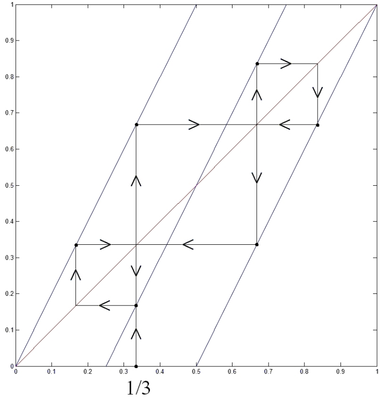
- The set of successors produced by x is given by

$$B^\infty(x) = \bigcup_{n=0}^{\infty} B^n(x).$$

When is $B^\infty(x)$ a finite set?

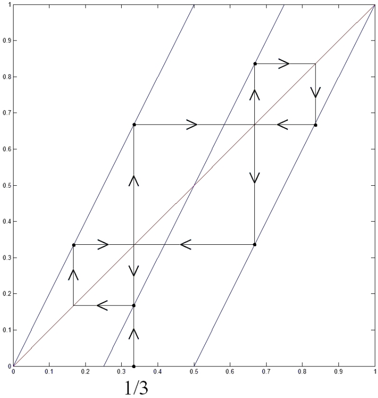
Finite orbits in branching dynamical systems

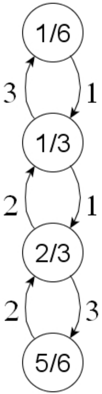
$$g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \tfrac{1}{2}.$$



Finite orbits in branching dynamical systems

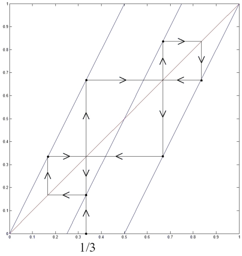
$$g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \tfrac{1}{2}.$$

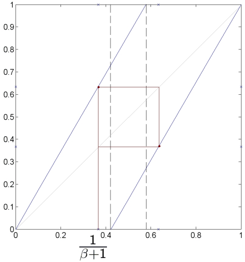




$$B^{\infty}(\tfrac{1}{3}) = \{\tfrac{1}{6}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{5}{6}\}$$

Related work on β-expansions and Bernoulli convolutions





Study of $\frac{\log |B^n(x)|}{n}$ for $n \to \infty$.

- De-Jun Feng and Nikita Sidorov
  2011 (Monatsh. Math.)
- Simon Baker
  2012 (arXiv: 1208.6195v1)
- Tom Kempton
  2012 (preprint)

Finite orbits in linear branching dynamical systems

Theorem

Let $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R}, j = 1, \ldots, k\}$ *a BDS with*

$$g_j(x) = \beta^{d_j} x + z_j, \quad \beta > 1,\ d_j \in \mathbb{N},\ z_j \in \mathbb{R}.$$

Finite orbits in linear branching dynamical systems

Theorem

Let $B = \{g_j : I_j \to I \mid I_j \subset I \subset \mathbb{R}, j = 1, \ldots, k\}$ *a BDS with*

$$g_j(x) = \beta^{d_j} x + z_j, \quad \beta > 1,\ d_j \in \mathbb{N},\ z_j \in \mathbb{R}.$$

If

- β *is a Pisot number (algebraic integer, whose conjugates are less than 1 in modulus) and*
- $z_j \in \mathbb{Q}(\beta)$,

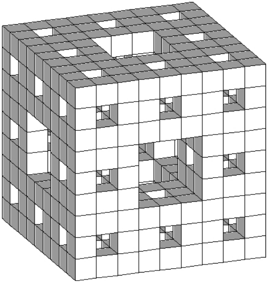
then $B^\infty(x)$ *is a finite set for all* $x \in \mathbb{Q}(\beta)$.

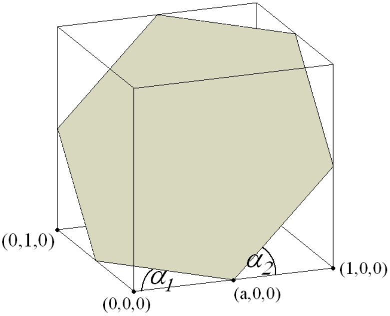
Special case:

Klaus Schmidt, 1980 (Bull. London Math. Soc.).

Slices and BDS

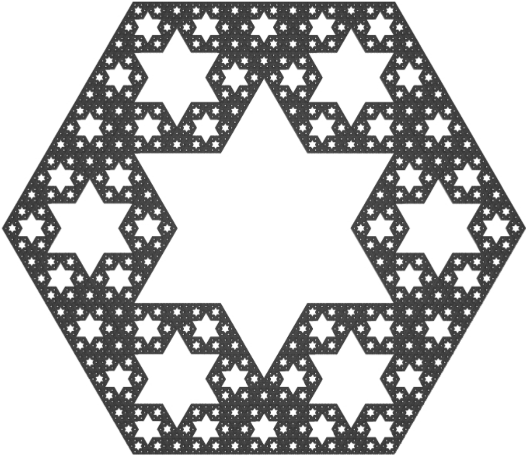
A slice is an intersection of a self-similar set and a hyperplane in $\mathbb{R}^n$.





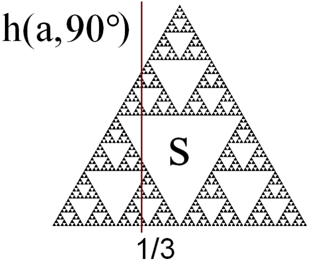
For the hyperplane holds: $H = H(a, \alpha_1, \ldots, \alpha_{n-1})$.

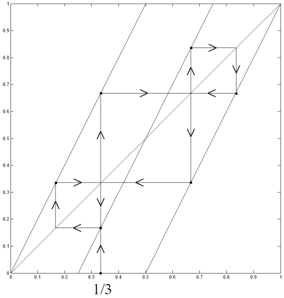
Slices and BDS

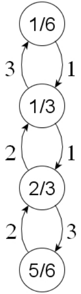


Proposition: Orthogonal slices through Sierpinski gasket.

- Consider the BDS consisting of the mappings
  $g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2},$
  which are surjections on $[0, 1]$,
- and the graph describing the orbit of a.

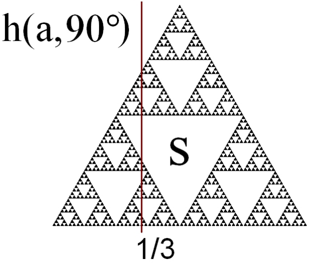


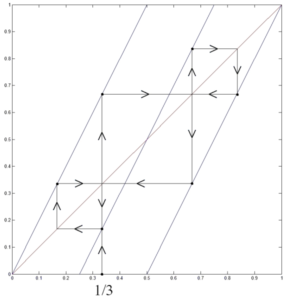


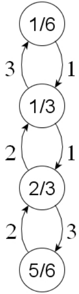


Proposition: Orthogonal slices through Sierpinski gasket.

- Consider the BDS consisting of the mappings
  $g_1(x) = 2x, \quad g_2(x) = 2x - 1, \quad g_3(x) = 2x - \frac{1}{2},$
  which are surjections on $[0, 1]$,
- and the graph describing the orbit of a.



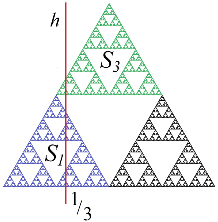




If $B^{\infty}(a)$ is finite, the graph is the Mauldin-Williams graph of $h \cap S$.

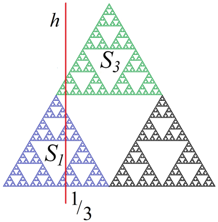
Slices and branching systems

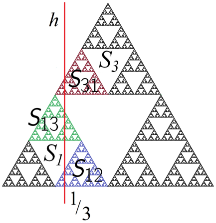
$S_i = f_i(S)$, $i = 1, 2, 3$. The BDS produces intercepts of lines.



Slices and branching systems

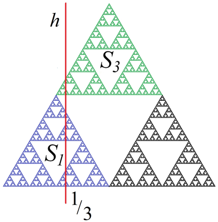
$S_i = f_i(S)$, $i = 1, 2, 3$. The BDS produces intercepts of lines.

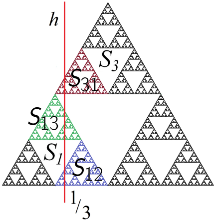


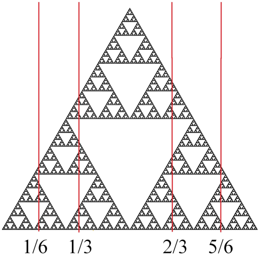


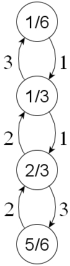
Slices and branching systems

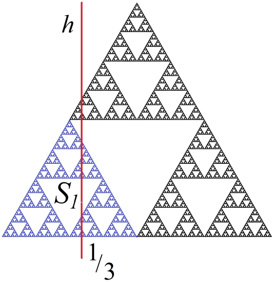
$S_i = f_i(S)$, $i = 1, 2, 3$. The BDS produces intercepts of lines.

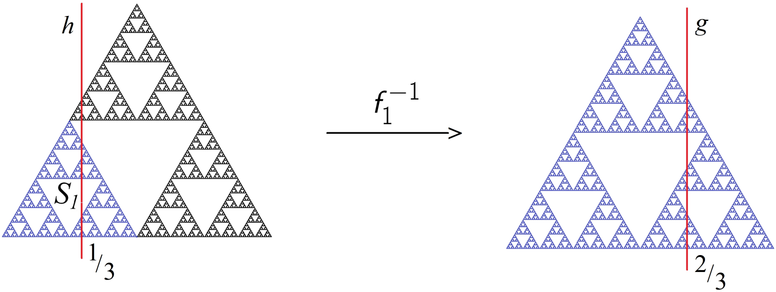


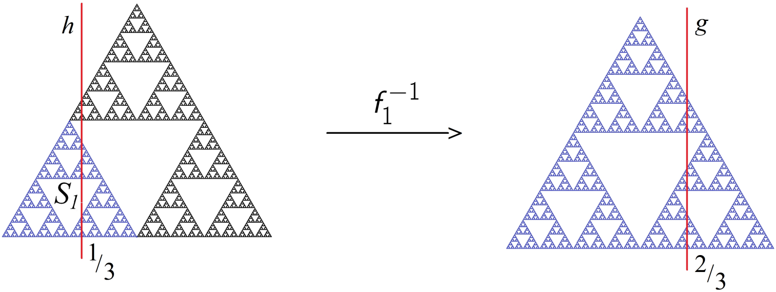




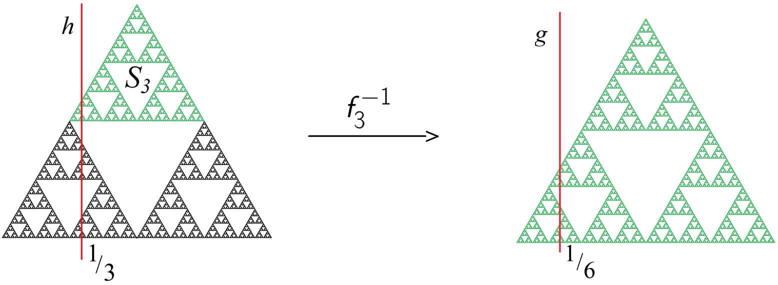



h
S1
1/3

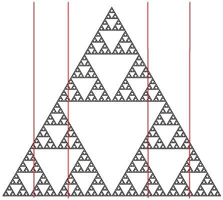

h
S1
1/3
f1^−1
g
2/3



$$g_1(x) = 2x$$



$$g_3(x) = 2x - \frac{1}{2}$$



Related work:

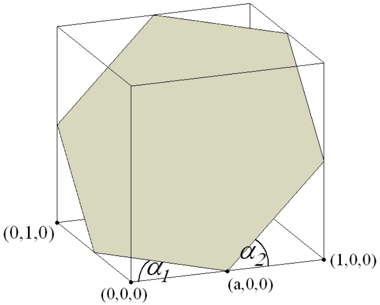
- Slicing the Sierpiński Gasket
  Balás Bárány, Andrew Ferguson, Károly Simon
  2011 (preprint)
- Dimension of Slices through the Sierpinski Carpet
  Anthony Manning, Károly Simon
  2010 (appears in TAMS)
- On the Dimensions of Sections through the Graph-diricted Sets
  Zhi-Ying Wu, Li-Feng Xi
  2010 (Ann. Acad. Scient. Finnicæ Math., Vol. 35)

Slices and BDS

- Let the self-similar set F be given by

$$f_j(x) = \frac{1}{\beta_j}(x + v_j), \quad \beta_j > 1, \; v_j \in \mathbb{R}^n$$

- and $H(a, \alpha_1, \ldots, \alpha_{n-1})$ a hyperplane intersecting F.



Slices and BDS

- Let the self-similar set F be given by

$$f_j(x) = \frac{1}{\beta_j}(x + v_j), \quad \beta_j > 1,\ v_j \in \mathbb{R}^n$$

- and $H(a, \alpha_1, \ldots, \alpha_{n-1})$ a hyperplane intersecting F.

Then the maps of the BDS producing the graph of intersection are given by

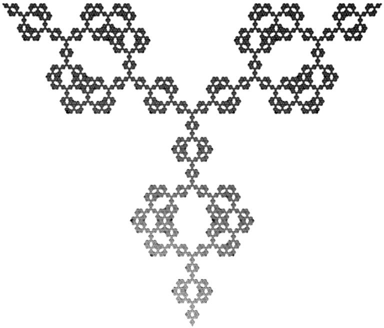
$$g_j(x) = \beta_j x + \langle \begin{pmatrix} -1 \\ \cot \alpha_1 \\ \vdots \\ \cot \alpha_{n-1} \end{pmatrix}, v_j \rangle$$

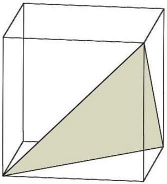
and the vertex set is given by $B^\infty(a)$.

Slices of finite type

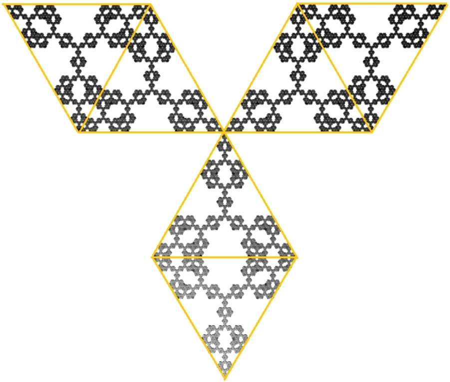
Definition

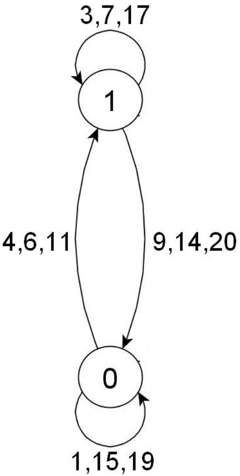
A slice is of finite type if the corresponding graph possesses finite many nodes ($\Leftrightarrow B^{\infty}(a)$ is a finite set.)



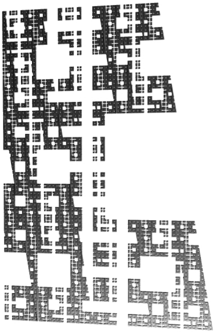


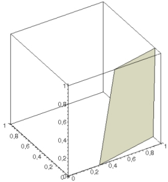
Slices of finite type

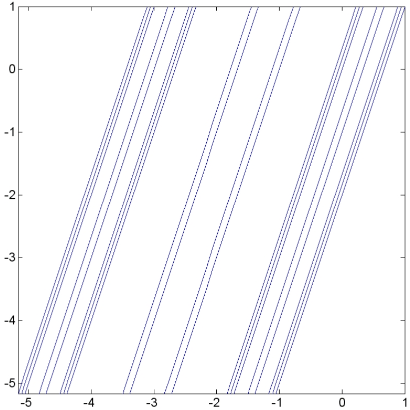




Slices of finite type





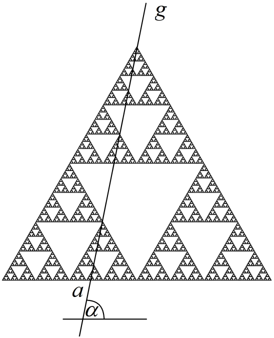


Proposition (Slices through Sierpinski gasket)

The slice $g(a, \alpha) \cap S$ *is of finite type*

$$\Leftrightarrow$$

The numbers $\frac{\sqrt{3}}{2} \cot \alpha$ *and* a *are rational.*



Theorem (Sufficient conditions for Pisot-fractals)

- *Let* $F \subset \mathbb{R}^n$ *a self-similar set given by*

$$f_j(x) = \beta^{-d_j}(x + v_j), \quad \textit{where } \beta > 1,\ d_j \in \mathbb{N},\ v_j \in \mathbb{R}^n$$

- *and let* $H(a, \alpha_1, ..., \alpha_{n-1})$ *a hyperplane intersecting* F*.*

Theorem (Sufficient conditions for Pisot-fractals)

- *Let $F \subset \mathbb{R}^n$ a self-similar set given by*

$$f_j(x) = \beta^{-d_j}(x + v_j), \quad \textit{where } \beta > 1,\ d_j \in \mathbb{N},\ v_j \in \mathbb{R}^n$$

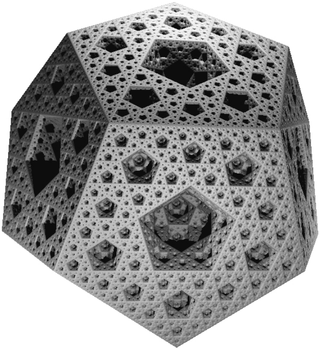
- *and let $H(a, \alpha_1, ..., \alpha_{n-1})$ a hyperplane intersecting F.*

Assume that the following conditions are fulfilled:

- *β is a Pisot number,*
- $\left\langle \begin{pmatrix} -1 \\ \cot \alpha_1 \\ \vdots \\ \cot \alpha_{n-1} \end{pmatrix}, v_j \right\rangle \in \mathbb{Q}(\beta) \quad \forall j,$
- $a \in \mathbb{Q}(\beta)$.

Then the slice $H \cap F$ is of finite type.

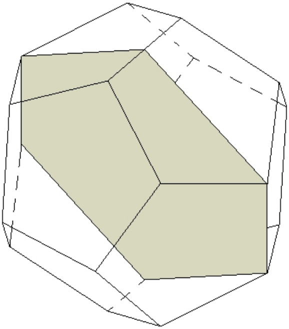
The golden dodecahedron (Mai The Duy, 2011)



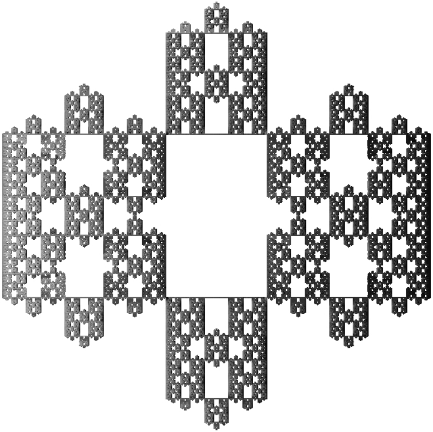
50 maps with overlaps,
2 scaling factors

Generated with "IFS Builder 3d v. 1.7.6", A. Kravchenko, D. Mekhontsev, Novosibirsk State University, (C) 1999-2011

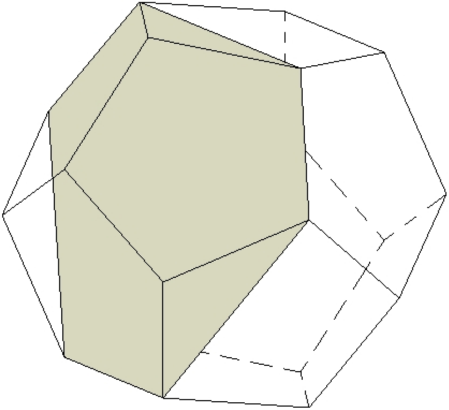
The golden dodecahedron



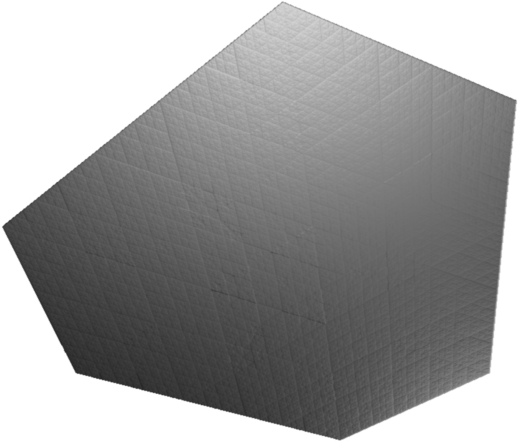
The golden dodecahedron



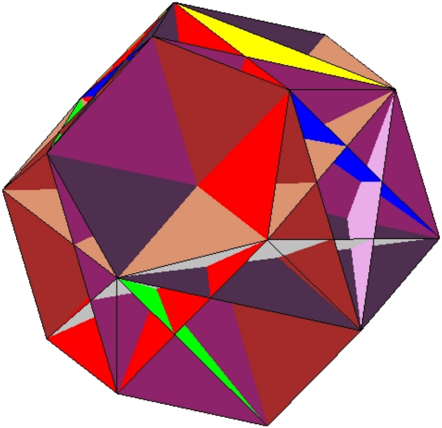
The golden dodecahedron



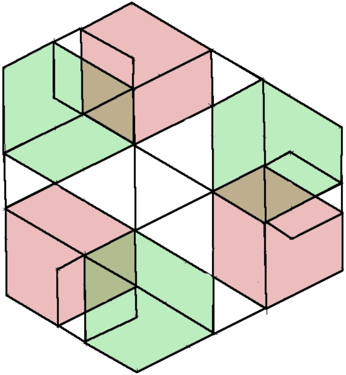
The golden dodecahedron



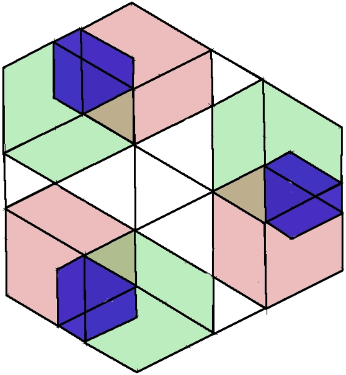
The golden dodecahedron



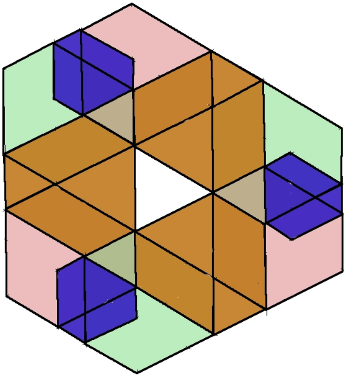
The golden dodecahedron



The golden dodecahedron



The golden dodecahedron



The golden dodecahedron