Approximation of fractals by tubular neighborhoods - geometric and analytic properties

M. Zähle

Friedrich Schiller University Jena

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1. Lipschitz-Killing curvature measures in classical (singular) curvature theory in $\mathbb{R}^d$

Notation

$$C_k(K,\cdot)\,, \quad k=0,\ldots,d$$

total curvatures: $C_k(K)=C_k(K,\mathbb{R}^d)$

Special cases $k=0$: total Gauss curvature = Euler characteristic,
$k=d$: volume (for completeness)

Convex geometry *(Steiner, Minkowski, Hadwiger, Santalo, ..., Groemer, Schneider)*
$C_k(K)$ kth intrinsic volume of a (poly)convex body K

Differential and integral geometry *(Weyl, Chern, Blaschke, Santalo, ..., Wintgen, Cheeger/Müller/Schrader)*
$C_k(K,\cdot)$ in terms of integrating the traces of powers of the Riemannian curvature tensor over a C^2-manifold K and integrating the higher order mean curvatures over the boundary ∂K

Special cases

if $K = M_m$ compact m-dimensional C^2-submanifold: $C_{m-k}(M_m)$ total k-th order mean curvature of M_m, $k = 2$ scalar curvature

if K smooth domain in $\mathbb{R}^d$ with boundary ∂K: $C_{d-2}(K)$ total mean curvature of ∂K

In general, the $C_k(K)$ arise as coefficients in the so-called Steiner (resp. Weyl) polynomial for the volume of parallel sets of small distances:

$$V(K_r) = \sum_{k=0}^{d} \mathrm{const}(d,k) C_{d-k}(K) r^k \,,$$

moreover, they form a complete system of certain Euclidean invariants (Hadwiger 1958, Z. 1990).

Relationships to spectral analysis:

$0 \geq \lambda_{1,l} \geq \lambda_{2,l} \geq \ldots$ eigenvalues of the Laplace operator Δ_l of M_m acting on l-forms, then $\operatorname{tr} e^{t\Delta_l} = \int_{M_m} p_t^l(x,x) d\mathcal{H}^m$

$$= \sum_{n=1}^{\infty} \exp(\lambda_{n,l} t) \sim (4\pi t)^{-m/2} \sum_{k=0}^{[m/2]} A_{k,l}(M_m) t^k + O(t^{1/2}), \quad t \downarrow 0$$

where $A_{k,l}(M_m)$ are the integrals over M_m of invariant polynomials of order $2k$ in the derivatives of the Riemannian metric
(Weyl, Minakshisundaram, Pleijel, Kac, McKean/Singer, **Patodi** (1971, general version which holds also locally)

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Donelli (1975, basing on a result of Patodi): for known constants $\gamma(k,l,m)$,

$$C_{m-2k}(M_m) = \sum_{l=0}^{2k} \gamma(k,l,m) A_{k,l}(M_m)$$

in particular, $A_{0,0}(M_m) = C_m(M_m)$ (Riemannian volume of M_m) and $A_{1,0}(M_m) = \frac{1}{3} C_{m-2}(M_m)$ (total scalar curvature)

Geometric measure theory - extension of the above geometric approaches *([Federer 1959], explicit representation [Z. 1986])*

k-th order curvature-direction measure as integral of kth generalized mean curvatures over the unit normal bundle $\mathrm{nor}K \subset \mathbb{R}^d \times S^{d-1}$ of a set K with positive reach (unique foot point property)

$$\widetilde{C}_k(K,\cdot) := \int_{\mathrm{nor}K\cap(\cdot)} S_{d-1-k}(\varkappa_1,\ldots,\varkappa_{d-1})\, d\mathcal{H}^{d-1}$$

with marginal $C_k(K,\cdot) := \widetilde{C}_k(K,(\cdot)\times S^{d-1})$ kth Lipschitz-Killing curvature measure on $\mathbb{R}^d$, $k=0,\ldots,d-1$, where

$$S_l((\varkappa_1,\ldots,\varkappa_{d-1}) := \mathrm{const}(d,l)\prod_{i=1}^{d-1}(1+\varkappa_i^2)^{-1/2}\sum_{1\le i_1\ldots\le i_l\le d-1}\varkappa_1\ldots\varkappa_l$$

lth symmetric function of generalized principal curvatures $-\infty < \varkappa_1(x,n) \le \varkappa_2(x,n)\ldots \le \varkappa_{d-1}(x,n) \le \infty$ on $\mathrm{nor}\,K$, (where $\infty(1+\infty^2)^{-1/2} =: 1$)

For $\varepsilon > 0$ and $K \subset \mathbb{R}^d$ recall

$$K_\varepsilon := \{x \in \mathbb{R}^d : \mathrm{dist}(x, K) \leq \varepsilon\}.$$

Theorem (Fu 1985)

For any compact $K \subset \mathbb{R}^d$ with $d \leq 3$, Lebesgue-a.e. $\varepsilon > 0$ is a regular value of the distance function of K and, hence, the closure of the complement of the the parallel set K_ε has positive reach.

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For arbitrary d and compact K with this property define the kth Lipschitz-Killing curvature measure of the parallel sets K_ε for such ε by

$$C_k(K_\varepsilon, \cdot) := (-1)^{d-1-k} C_k\left(\overline{(K_\varepsilon)^c}, \cdot\right)$$

(consistent definition).

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For classical sets K as above we have

$$(w) \lim_{\varepsilon \to 0} C_k(K_\varepsilon, \cdot) = C_k(K, \cdot),$$

for fractal sets explosion. Therefore rescaling is necessary:

2. Fractal curvatures - approximation by close neighborhoods

(References below)

F self-similar (random) set in $\mathbb{R}^d$ with Hausdorff dimension D satisfying (S)OSC

Under the additional assumption on the regularity of the neighborhoods F_ε and some integrability condition the following limits exist (almost surely):

$$C_k^{frac}(F) := \lim_{\varepsilon\to 0} \varepsilon^{D-k} C_k(F_\varepsilon)$$

in the "non-arithmetic case" and

$$C_k^{frac}(F) := \lim_{\delta\to 0} \frac{1}{|\ln\delta|} \int_\delta^1 \varepsilon^{D-k} C_k(F_\varepsilon)\, \frac{1}{\varepsilon}\, d\varepsilon\,.$$

in general.

(Integral representation for $C_k(F)$ which admits some explicit or numerical calculations.)

Curvature-direction measure version (deterministic case):

$$\begin{aligned}\widetilde{C}_k^{frac}(F,\cdot) : &= (w)\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_\delta^1 \varepsilon^{D-k}\widetilde{C}_k(F_\varepsilon,\cdot)\,\frac{1}{\varepsilon}\,d\varepsilon \\ &= C_k(F)\big(\mathcal{H}^D(F)^{-1}\,\mathcal{H}^D|_F\times\mathcal{D}_k^F\big)(\cdot)\,. \\ &= (w)\lim_{\varepsilon\to 0}\varepsilon^{D-k}\widetilde{C}_k(F_\varepsilon,\cdot)\end{aligned}$$

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Interpretation of the factors $C_k(F)\mathcal{H}^D(F)^{-1}$: some fractal analogues of the higher order **pointwise mean curvatures** on smooth submanifolds, here: constant values because of self-similarity,
$\mathcal{D}_k^F$ distributions on the unit sphere in $\mathbb{R}^d$ measuring the **anisotropy** of F "weighted by these mean curvatures",

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Main tool and additional result: interpretation of the above factors as curvature densities, permits to consider other types of (random) fractals with scaling properties:

3. Average curvature densities

Let O be from (SOSC), $SO := \bigcup_{i=1}^{N} S_i O$ for the generating similarities $S_1, \dots, S_N$ with contraction rations $r_1, \dots, r_N$.

For $a > 1$, $\varepsilon_0 > 0$ and $b := \max(2a, \varepsilon_0^{-1}|O|)$ let $\{A_F(x,\varepsilon) : x \in F,\, 0 < \varepsilon < \varepsilon_0, \ \}$ be a locally homogeneous neighborhood net:

$A_F(x,\varepsilon) \subset F_\varepsilon \cap B(x, a\varepsilon)$ and

$A_F(x,\varepsilon) = S_i(A_F(S_i^{-1}x, r_i^{-1}\varepsilon))$ if $x \in S_i F$ and $\varepsilon < b^{-1} d(x, \partial S_i(O))$ (homogeneity).

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Examples:

1. $A_F(x, \varepsilon) = F_\varepsilon \cap B(x, a\varepsilon)$
2. $A_F(x, \varepsilon) = F_\varepsilon \cap \Pi_F^{-1}\big(B(x, \varepsilon)\big)$,
   the set of those points from F_ε which have a foot point on F within the ball $B(x, \varepsilon)$
3. $A_F(x, \varepsilon) = \{y \in F_\varepsilon : |y - x| < \varrho_F(y, \varepsilon)\}$,
   where $\varrho_F(y, \varepsilon)$ is determined by $\mathcal{H}^D\big(F \cap B(y, \varrho_F(y, \varepsilon))\big) = \varepsilon^D$

Fractal curvature densities:

For $\mathcal{H}^D$-a.a. $x \in F$ the following limit exists

$$\lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(x,\varepsilon)\big) \frac{1}{\varepsilon} \, d\varepsilon$$

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and equals the constant

$$\mathcal{H}^D(F)^{-1} \big(\sum_{i=1}^N r_i^D |\ln r_i| \big)^{-1} \int_F \int_{\frac{d(y,\partial(SO))}{2a}}^{\frac{d(y,\partial O)}{2a}} \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(y,\varepsilon)\big) \frac{1}{\varepsilon} \, d\varepsilon \, \mathcal{H}^D(dy)$$

provided the last double integral converges.

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The limit agrees with the former local variant $C_k(F)\mathcal{H}^D(F)^{-1}$ if the sets $A_F(x,\varepsilon)$ are chosen as in Example 3. ($k = 0, \ldots, d$.)

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Analogous result for self-similar random sets F can be proved.

References and extensions

Earlier and recent literature for the special case of the Minkowski content: Lapidus, Falconer, Gatzouras, Lapidus/Pearse/Winter, Rataj/Winter, Kesseböhmer/Kombrink, Freiberg/Kombrink, Kombrink, ...

Self-similar sets:

S. Winter, *Curvature measures and fractals*, Dissertationes Math. **453**, 2008.
(deterministic self-similar sets with polyconvex neighborhoods, curvature measures)

M. Zähle: *Lipschitz-Killing curvatures of self-similar random fractals*, TAMS **363**, 2011.
(self-similar random sets with singular neighborhoods, total curvatures)

S.Winter, M. Zähle: *Fractal curvature measures of self-similar sets*, Adv. in Geom. (to appear).
(deterministic measure version for singular neighborhoods)

J. Rataj, M. Zähle: *Curvature densities of self-similar sets*, Indiana Univ. Math. J. (to appear).
(dynamical approach to local and global curvatures, average versions)

T. Bohl, M. Zähle: *Curvature-direction measures of self-similar sets*, Geometria Dedic. (to appear)
(extension of the average limits to non-isotropic quantities, new short proof for the measure versions)
M. Zähle: *Curvature measures of fractal sets*, Contemp. Math.
(survey on previous results and new short proof for ordinary limits)

Self-conformal sets:

T. Bohl: *Fractal curvatures and Minkowski content of self-conformal sets*, arxiv.org/abs/1211.3421 (Thesis, University of Jena)
(extends all former results on average curvature (direction) measures to self-conformal sets, more involved tools from the theory of dynamical systems)

4. Related Dirichlet forms for the case of the Sierpinski gasket

$F := G$ Sierpinski gasket in $\mathbb{R}^d$ with Hausdorff dimension $d_H = \ln(d+1)/\ln 2$ and walk dimension $d_W = \ln(d+3)/\ln 2$.

Consider the special Dirichlet forms on the parallel sets w.r.t. $L_2(G_\varepsilon)$

$$\mathcal{E}_\varepsilon(f) := \int_{G_\varepsilon} |\nabla f(x)|^2 \, dx$$

with Neumann boundary conditions and domain $H^1_{(N)}(G_\varepsilon))$ together with the known Dirichlet form $\mathcal{E}$ on the gasket with domain $\mathrm{Lip}(\frac{d_W}{2}, 2, \infty)$.

Then we get for any family $f_\varepsilon \in dom(\mathcal{E}_\varepsilon)$ with $tr|_G f_\varepsilon = f \in dom(\mathcal{E})$,

$$\liminf_{\delta \to 0} \frac{c(d)}{|\ln \delta|} \int_\delta^1 \varepsilon^{-(d_W - 2 + d - d_H)} \mathcal{E}_\varepsilon(f_\varepsilon) \frac{1}{\varepsilon} \, d\varepsilon \geq \mathcal{E}(f)$$

and we obtain such a family for which the limit exists and is equal to the right hand side.