Ruelle Operator with Weakly Contractive Iterated Function Systems

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Introduction

Symbolic system

Let $\Sigma = \{1, \cdots N\}^{\mathbb{N}}$ be the one-sided symbolic space and let

$$\sigma : \omega = i_0 i_1 \cdots i_{n-1} \cdots \to \sigma(\omega) = i_1 \cdots i_{n-1} \cdots$$

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For any $x = (x_1 x_2 \cdots), y = (y_1 y_2 \cdots) \in \Sigma$, let

$$d(x, y) = \frac{1}{k+1} \quad \text{if } x_i = y_i \text{ for all } i < k \text{ and } x_k \neq y_k.$$

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- (Σ, d) is a compact metric space.

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Ruelle operator

The *Ruelle operator* $\mathcal{T} : C(\Sigma) \to C(\Sigma)$ is defined as

$$\mathcal{T} f(x) = \sum_{y \in \sigma^{-1}(x)} e^{\phi(y)} f(y), \qquad f \in C(\Sigma).$$

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Theorem (see Bowen 1975)

Let ϕ be a Hölder continuous function on Σ. Then
(i) *ϱ is the unique positive simple maximal eigenvalue of $\mathcal{T}$ acting on the space of all Hölder continuous functions on Σ;*
(ii) *there exists a unique positive eigenfunction $h \in C(\Sigma)$ and a unique probability eigenmeasure $\mu \in C^*(\Sigma)$ such that*

$$\mathcal{T}h = \varrho h, \qquad \mathcal{T}^*\mu = \varrho\mu, \qquad \langle \mu, h \rangle = 1;$$

(iii) *for any $f \in C(\Sigma)$, $\varrho^{-n}\mathcal{T}^n(f)$ converges uniformly to a constant multiple of h.*

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- This is called the Ruelle operator theorem for symbolic system (Σ, ϕ).

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Theorem (Walters 1975)

If

$$\sum_{n=1}^{\infty} var_n(\phi) < \infty,$$

then there exists a unique positive eigenfunction $h \in C(\Sigma)$ and a unique probability eigenmeasure $\mu \in C^*(\Sigma)$ such that

$$\mathcal{T}h = \varrho h, \qquad \mathcal{T}^*\mu = \varrho\mu, \qquad \langle \mu, h \rangle = 1.$$

Moreover, for any $f \in C(\Sigma)$, $\varrho^{-n}\mathcal{T}^n(f)$ converges uniformly to a constant multiple of h.

Iterated function system

Let X be a non-empty convex compact subset of $\mathbb{R}^d$, let $\{w_j\}_{j=1}^m$ be a set of maps from X into X.

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Contractive IFS

The iterated function system (IFS) $\{w_j\}_{j=1}^m$ is called *contractive*, if there is a $0 < a < 1$ such that for any $1 \leq j \leq m$

$$\sup_{|x-y|\leq t} |w_j(x) - w_j(y)| < at \quad \text{for all } t > 0.$$

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Weakly contractive IFS

The IFS $\{w_j\}_{j=1}^m$ is called *weakly contractive*, if for any $1 \le j \le m$

$$\sup_{|x-y|\le t} |w_j(x) - w_j(y)| < t \quad \text{for all } t > 0.$$

Iterated function system

Invariant set (Hata 1985)

Let $\{w_j\}_{j=1}^m$ be a weakly contractive IFS defined on convex compact set $X(\subset \mathbb{R}^d)$. There exists a non-empty compact set K such that $K = \bigcup_{j=1}^m w_j(K)$. We call K the invariant set of the IFS $(X, \{w_j\}_{j=1}^m)$.

Ruelle operator

Ruelle operator

With each w_j, we associate a positive continuous function p_j as a weight function (or potential function). We can set up the *Ruelle operator* as follows on the space $C(K)$ of real continuous functions on K:

$$T(f)(x) = \sum_{j=1}^{m} p_j(x) f(w_j(x)), \qquad f \in C(K).$$

Dini continuous

Let p be a function defined on X. Define

$$\alpha_p(t) = \sup_{|x-y|\leq t} |p(x) - p(y)|.$$

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The function p is called *Dini continuous* if

$$\int_0^1 \frac{\alpha_p(t)}{t} dt < \infty.$$

Ruelle operator theorem

Spectral radius

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Definition

We say that the Ruelle operator theorem holds for the system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$, if there exists a unique positive function $h \in C(K)$ and a unique probability measure $\mu \in M(K)$ such that

$$Th = \varrho h, \qquad T^*\mu = \varrho\mu, \qquad \langle \mu, h \rangle = 1.$$

Moreover, for every $f \in C(K)$, $\varrho^{-n}T^n f$ converges to $\langle \mu, f \rangle h$ in the supremum norm, and for every $\xi \in M(K)$, $\varrho^{-n}T^{*n}\xi$ converges weakly to $\langle \xi, h \rangle \mu$.

Results known

Theorem (Fan and Lau 1999 JMMA)

If the IFS $\{w_j\}_{j=1}^m$ is contractive and each p_j is Dini continuous, then the Ruelle operator theorem holds for the system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$.

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Application (Fan, Lau, Rao and Ye 2001)

Let $\{w_j\}_{j=1}^m$ be a contractive conformal IFS. Then, both OSC and SOSC are equivalent to $0 < \mathcal{H}^\alpha(K) < \infty$.

Results known

Theorem (Lau and Ye 2001 Studia Math)

Let $\{w_j\}_{j=1}^m$ be a weakly contractive IFS and let each p_j be Dini continuous. If

$$\sup_{x\in K}\sum_{j=1}^m p_j(x)\sup_{y\neq z}\frac{|w_j(y)-w_j(z)|}{|y-z|}<\varrho,$$

then the Ruelle operator theorem holds for the system $(X,\{w_j\}_{j=1}^m,\{p_j\}_{j=1}^m)$.

Results known

Theorem (Jiang and Ye 2010 ETDS)

If

$$\sup_{x\in K}\sum_{j=1}^{m} p_j(x)\sup_{y\neq x}\frac{|w_j(x)-w_j(y)|}{|x-y|}<\varrho,$$

then the Ruelle operator theorem holds for the system $(X,\{w_j\}_{j=1}^m,\{p_j\}_{j=1}^m)$.

Problem

Question

Professor Jiang asked, in a private communication, if the Ruelle operator theorem holds for a system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ satisfying the condition:

$$\sum_{j=1}^{m} p_j(x) \cdot |w_j'(x)| < \varrho \quad \text{for all } x \in X. \tag{1}$$

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Remark

It seems that the condition (1) is more natural than the condition given by paper of Jiang and Ye.

Main results

Definition

Let X be a non-empty convex compact subset of $\mathbb{R}^d$. We call $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ a Dini (or Lipschitz) system, if all maps $w_j : X \to X,\ 1 \leq j \leq m$, are both continuously differentiable and weakly contractive and, all potentials $p_j : X \to \mathbb{R}^+,\ 1 \leq j \leq m$, are positive Dini (or Lipschitz) continuous.

Main results

Theorem

Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Dini system with $w'_j(x) \neq 0$ for all j. Suppose that the condition (1) is satisfied. Then the Ruelle operator theorem holds for this Dini system.

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Corollary

If

$$\max_{x \in X} \sum_{j=1}^m p_j(x) \cdot |w_j'(x)| < \min_{x \in X} \sum_{j=1}^m p_j(x),$$

then the Ruelle operator theorem holds for this Dini system.

Main results

Theorem

Let $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ be a Lipschitz system. Suppose that the condition (1) is satisfied. Then $\varrho_e(T) < \varrho(T)$. Moreover, for any Lipschitz continuous function f defined on K, the sequence $\varrho^{-n} T^n f$ converges with a specific geometric rate.

Remark

Theorem (Barnsley *et al* 1988)

Let the Dini system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ *satisfy the conditions:* $\sum_{j=1}^m p_j(x) = 1$, *and*

$$\sup_{y \neq x} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x-y|} < 1.$$

Then the Ruelle operator theorem holds.

Remark

Theorem (Hennion 1993)

Let the Lipschitz system $(X, \{w_j\}_{j=1}^m, \{p_j\}_{j=1}^m)$ satisfy the condition:

$$\sup_{\substack{x,y\in X \\ y\neq x}} \sum_{j=1}^{m} p_j(x) \cdot \frac{|w_j(x) - w_j(y)|}{|x-y|} < \varrho.$$

Then, the operator P is quasi-compact, i.e., $\varrho_e(P) < \varrho(P)$.

Remark

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Note that for any $x \in K$,

$$|w_j'(x)| \leq \sup_{y \neq x} \frac{|w_j(x) - w_j(y)|}{|x - y|}.$$

We see that the above results have been generalized.

Thank you!