Three-dimensional flows: several phenomena

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Hongkong, Dec. 11th, 2012

Setting

M compact Riemannian manifold without boundary

X C^1 vector field on M

ϕ_t flow generated by X

$\Phi_t = d\phi_t$: the derivative w.r.t. the space variable

$\sigma \in M$ is a *singularity* if $X(\sigma) = 0$. Other points are called *regular points*.

dynamics of ϕ_t: limit behaviors when $t \to \pm\infty$ (study from Poincaré, Lyapunov)

Filtration

A filtration of X is a collection of finite neighborhoods

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M,$$

such that

$$\phi_t(\overline{U_i}) \subset \mathrm{Int}(U_i), \quad \forall i, \ \forall t > 0.$$

Denote the maximal invariant set

$$\Lambda_i = \bigcap_{t \in \mathbb{R}} \phi_t(U_i \setminus U_{i-1}).$$

Morse-Smale

X is *Morse-Smale* if X admits a filtration

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M,$$

such that each Λ_i is a hyperbolic periodic orbit or a hyperbolic singularity.

Morse-Smale systems are simple stable dynamics.

Peixoto's result

No way to understand the dynamics of every vector field.

Peixoto's theorem

Assume that $\dim M = 2$. There is a dense open set $\mathcal{G} \subset \mathcal{X}^1(M)$ such that every $X \in \mathcal{G}$ is Morse-Smale.

Thus, sometimes we can understand the dynamics for "most" systems.

Hyperbolic sets

A compact invariant set Λ is *hyperbolic* such that there is an invariant splitting

$$T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u,$$

w.r.t. Φ_t, such that E^s is uniformly contracted, E^u is uniformly expanded.

An example of non-trivial hyperbolic set is a horseshoe or an Anosov flow.

Uniform hyperbolicity

X is uniformly hyperbolic if X has a filtration

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M,$$

such that each Λ_i is a transitive hyperbolic set.

$\dim M = 3$. A compact invariant set Λ is *singular hyperbolic* if (for X or $-X$) there is an invariant splitting

$$T_\Lambda M = E^s \oplus E^{cu}, \quad \dim E^s = 1$$

w.r.t. Φ_t, such that E^s is uniformly contracted, E^{cu} is area-expanded.

An example of non-trivial hyperbolic set is a hyperbolic set or a geometric Lorenz attractor.

Singular hyperbolicity

X is singular hyperbolic if X has a filtration

$$\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M,$$

such that each Λ_i is a transitive singular hyperbolic set, or a transitive hyperbolic set.

Next, we will talk about two semi-local phenomena.

Transverse homoclinic intersection

X has a *transverse homoclinic intersection* if X has a hyperbolic periodic orbit γ and $W^s(\gamma)$ intersects $W^u(\gamma)$ transversely at some point.

Properties

- Poincaré showed that transverse homoclinic intersections can survive under small perturbations. Moreover, if a system has one transverse homoclinic intersection, then it has infinitely many transverse homoclinic intersections.
- Birkhoff showed that if a plane system has one transverse homoclinic intersection, then it has infinitely many hyperbolic periodic orbits.
- Smale proved that the existence of transverse homoclinic intersection is equivalent to the existence of horseshoe.

Theorem, joint work with S. Gan

For every three dimensional vector field, in the C^1 topology, either it can be accumulated by Morse-Smale vector fields, or it can be accumulated by vector fields with a transverse homoclinic intersection.

Similar results were got for diffeomorphisms (the discrete case): Pujals-Smabarino (dim=2), Bonatti-Gan-Wen (dim=3), Crovisier (any dimension)

The difficult here is that we need to analysis some compact invariant set with singularities...

Another semi-local phenomenon is homoclinic tangency: X has a *homoclinic tangency* if X has a hyperbolic periodic orbit γ and $W^s(\gamma)$ intersects $W^u(\gamma)$ non-transversely at some point.

By Newhouse, homoclinic tangencies could happen “typically” in the C^2 topology.

For C^1 topology, we don’t know!

Theorem Arroyo-Hertz

Every three-dimensional non-singular vector field can be either accumulated by a uniformly hyperbolic vector field, or accumulated by ones with a homoclinic tangency.

The general case: we don't know!

Conjecture

Palis-Arroyo-Hertz-Morales-Pacifico

Every three-dimensional vector field can be either accumulated by a singular hyperbolic vector field (Lorenz-like, and contains uniformly hyperbolic ones), or accumulated by ones with a homoclinic tangency.

We are looking forward to finding a proof with S. Crovisier.

Partial result

Bonatti-Gan-Yang, Gan-Yang

For C^1 generic three dimensional vector field X, if the chain recurrent class of a singularity has a dominated splitting w.r.t. the tangent flow, then it is a singular hyperbolic attractor or repeller.