Conformal Invariance of the Exploration Path in 2D Critical Bond Percolation in the Square Lattice

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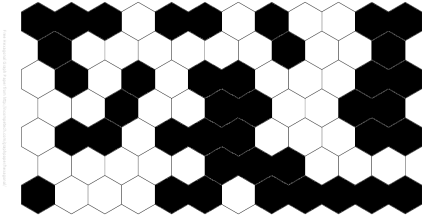
(Joint work with Jonathan TSAI (HKU) and Wang ZHOU(NUS))

Critical Site Percolation in the Hexagonal Lattice

For each site on the hexagonal lattice, we flip a fair coin.

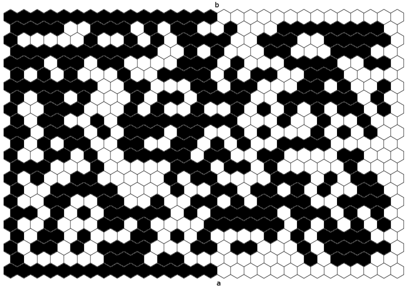
- Heads – we colour the site black.
- Tails – we colour the site white.



This is the *critical site percolation on the hexagonal lattice*.

The Site Percolation Exploration Path

We apply boundary conditions – black to the left, and white to the right – and flip coins for the other sites.



Then there is a path from a to b on the lattice that has black hexagons to its left and white hexagons to its right. This is the *site percolation exploration path*.

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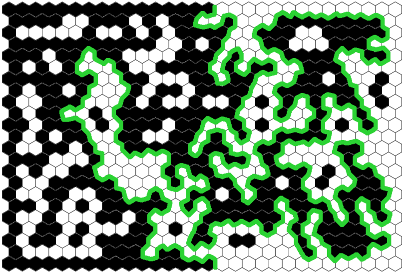

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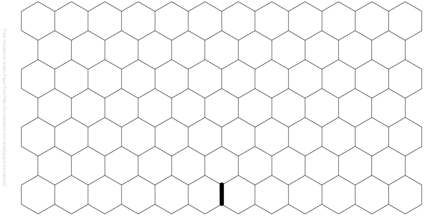
The Site Percolation Exploration Path (cont.)

Another way of constructing the path is as follows. At each step of the path, we flip a fair coin.

- Heads – the path turns right;
- Tails – the path turns left;

unless the path is forced to go in a particular direction.



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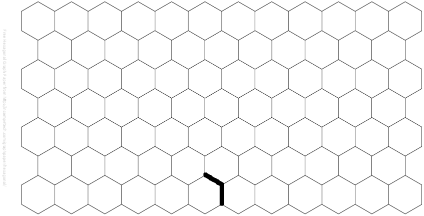

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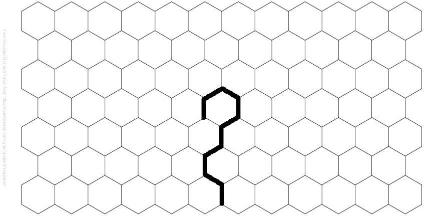


Theorem (Smirnov (2001), Camia and Newman (2007))

The scaling limit (i.e. the limit as the mesh-size of the lattice tends to zero) of the site percolation exploration path converges to stochastic (Schramm) Loewner evolution with parameter $\kappa = 6$ (SLE_6).

What is stochastic (Schramm) Loewner evolution?

- Invented by O. Schramm in 1999.
- Describes conformally invariant curves in the plane.
- Cardy's formula: the crossing probability of a percolation from an interval of an edge to that of another of an equilateral triangle.

The Loewner Transform

For some real-valued function $\xi : [0, \infty) \to \mathbb{R}$, the chordal Loewner differential equation is

$$\frac{\partial g}{\partial t}(z, t) = \frac{2}{g(z, t) - \xi(t)} \text{ with } g(z, 0) \equiv z$$

The solution $g_t(z) = g(z, t)$ is a conformal map of $H_t \subset \mathbb{H}$ onto $\mathbb{H}$ where $\mathbb{H} = \{z : \mathrm{Im}[z] > 0\}$ is the complex upper half-plane.
It is often the case that $H_t = \mathbb{H} \setminus \gamma[0, t]$ where γ is a curve in $\mathbb{H}$ starting from 0 and ending at ∞.

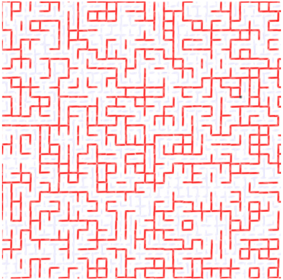
We can think of the chordal Loewner differential equation as defining a transform $\xi \mapsto \gamma$ which we call the *Loewner transform*. ξ is called the Loewner driving function of the curve γ.

Stochastic Loewner evolution with parameter κ is the Loewner transform of $\sqrt{\kappa}B_t$ where B_t is standard 1-d Brownian motion.

Critical Bond Percolation on the Square Lattice

How about on the square lattice $\mathbb{Z}^2$? For each edge in the lattice we flip a fair coin.

- Heads – we keep the edge.
- Tails – we delete the edge.

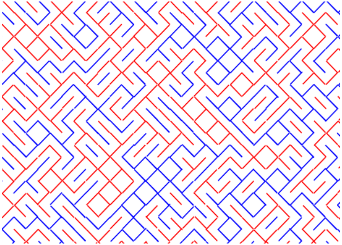


This is the *critical bond percolation on the square lattice*.

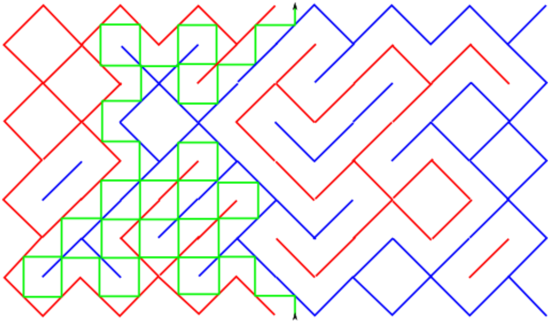
Critical Bond Percolation on the Square Lattice (cont.)

We can consider the same process on the dual lattice. For each site in the lattice we flip a fair coin.

- Heads – we add a diagonal edge from the top left vertex to the bottom right vertex.
- Tails – we add a diagonal edge from the bottom left vertex to the top right vertex.



We apply boundary conditions and flip coins for the other sites.



Then there is a rectilinear path from a to b on the original lattice that lies in the “corridor” between red and blue edges. This is the *bond percolation exploration path*.

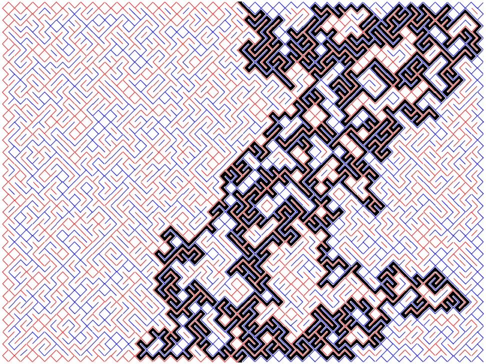
A Main Conjecture

- Whether the critical bond percolation exploration process on this square lattice converges to the trace of SLE_6 or not is an important conjecture in mathematical physics and probability.
- See P.5 in the document at "http://www.math.ubc.ca/ slade/newsletter.10.2.pdf" "... *But although site percolation on the triangular lattice is now well understood via SLE6, the critical behaviour of bond percolation on the square lattice, which is believed to be identical, is not at all understood from a mathematical point of view. Kenneth G. Wilson was awarded the 1982 Nobel Prize in Physics for his work on the renormalization group which led to an understanding of universality within theoretical physics. However, there is as yet no mathematically rigorous understanding of universality for two-dimensional critical phenomena* ..."

Main Theorem

Theorem

The scaling limit of the bond percolation exploration path converges to stochastic Loewner evolution with parameter $\kappa = 6$ (SLE_6).



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3. Show that the conditioned path has Loewner driving function that converges subsequentially to an ϵ-*semimartingale*, i.e. a martingale plus a finite $(1+\epsilon)$-variation process.

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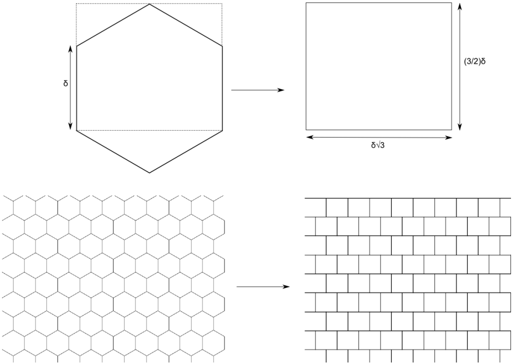
4. Exploit the locality property of bond percolation exploration path to show that the Loewner driving term of the bond percolation exploration path converges to $\sqrt{6}B_t$.

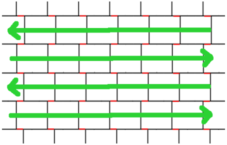
Idea of the Proof

1. Modify the lattice in order to "convert" the site percolation exploration path (on the hexagonal lattice) to a path on the rectangular lattice.
2. Apply a conditioning (restriction) procedure to this path to make it "close" to the bond percolation exploration path (on the square lattice).
3. Show that the conditioned path has Loewner driving function that converges subsequentially to an ϵ-*semimartingale*, i.e. a martingale plus a finite $(1+\epsilon)$-variation process.
4. Exploit the locality property of bond percolation exploration path to show that the Loewner driving term of the bond percolation exploration path converges to $\sqrt{6}B_t$.
5. Apply standard arguments to deduce that the scaling limit of the bond percolation exploration path converges to SLE_6.

By replacing the hexagonal sites in the hexagonal lattice with rectangles we convert the hexagonal lattice into a "brick-wall" lattice.

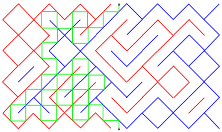


We then shift the rows of the brick-wall lattice left and right alternatively to get a rectangular lattice



This induces a path on the rectangular lattice which is in a 2δ-neighbourhood of the site percolation exploration path. This is the $+BP$ (Brick-wall Process). Similarly, by shifting the rows the other way, we get the $-BP$.

In particular, the $\pm$BP both converge to SLE_6 as the mesh-size δ tends to 0.

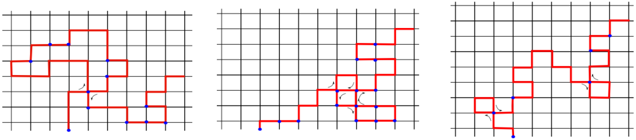


Note that the $+$BP can go in the same direction for two consecutive edges. The bond percolation path cannot. We will condition the $+$BP to not go in the same direction for two consecutive edges. However we do this in two steps.

First we define the free vertices of the $+$BP to be the vertices where there are 3 possible paths for the next vertex. We condition the $+$BP to not go in the same direction for two consecutive edges at the free vertices. This conditioned path is the $+$CBP (Conditioned Brick-wall Process). Similarly, we define the $-$CBP.

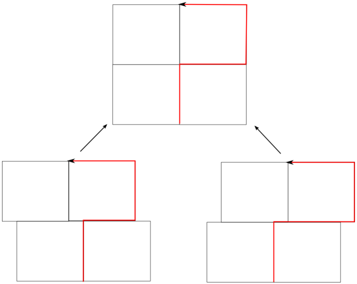
Idea of the Proof: Conditioning the $\pm$BP (cont.)

We then condition the $\pm$CBP at the non-free vertices as well to get the $\pm\partial$CBP (Boundary Conditioned Brick-wall Process).

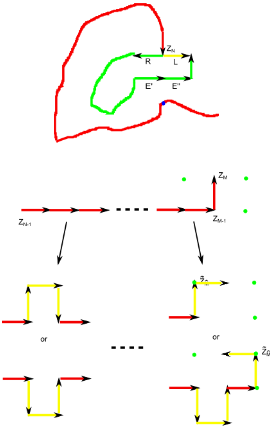


$+$BP $\quad$ $+$CBP $\quad$ $+\partial$CBP

It turns out that the bond percolation exploration path (on the square lattice) is topologically the same as alternate pastings of the $+\partial$CBP and $-\partial$CBP.



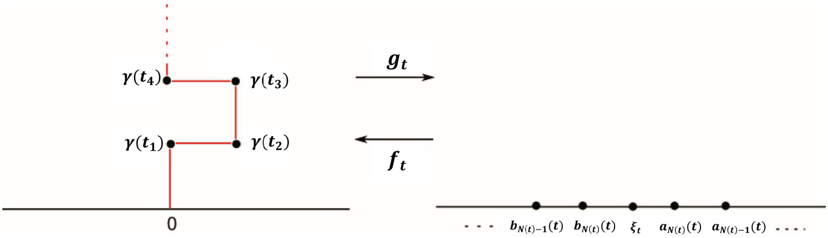
Also, we can couple the ±CBP and the ±∂CBP such that their respective Loewner driving functions are close.



Hence it is sufficient to study the +CBP.

For simplicity we work in $\mathbb{H}$. Using Schwarz-Christoffel transformation as in Tsai (2009), we can write the Loewner driving function of any path on the lattice as

$$\xi_t = \frac{1}{2}\left[a_1(t) + b_1(t) + \sum_{k=2}^{N(t)} L_k(a_k(t) - b_k(t))\right],$$



where L_k is $+1$ if the path turns right and -1 if the path turns left at the kth step.

Consider the +CBP.

We choose $0 = m_0 < m_1 < m_2 < \ldots$ random steps, defined recursively so that +BP has a definite increment in its half-plane capacity, adapted to the process appropriately.

Then if we let $M_n = \xi_{t_{m_n}}$, we have

$$M_n - M_{n-1} = R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}}) + \frac{1}{2} \sum_{k=m_{n-1}+1}^{m_n} L_k \big(a_k(t_{m_n}) - b_k(t_{m_n}) \big)$$

where

$$R_{n-1}(t) = \frac{1}{2} \Big[a_1(t) + b_1(t) + \sum_{k=2}^{m_{n-1}} L_k(a_k(t) - b_k(t)) \Big].$$

Idea of the Proof: Driving term convergence of the +CBP

Letting

$$\Delta_{j,n} = \left[(a_j(t_{m_n}) - a_{j+1}(t_{m_n})) - (b_j(t_{m_n}) - b_{j+1}(t_{m_n})\right],$$

we can telescope the above sum and take conditional expectations to get

$$\begin{aligned} \mathbb{E}\left[M_n - M_{n-1}|\mathcal{F}_{m_{n-1}}\right] &= \mathbb{E}\left[R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}})|\mathcal{F}_{m_{n-1}}\right] \\ &\quad + \frac{1}{2}\sum_{j=m_{n-1}+1}^{m_n} \mathbb{E}\big[\Delta_{j,n} \sum_{k=m_{n-1}+1}^{j} L_k|\mathcal{F}_{m_{n-1}}\big]. \end{aligned}$$

Using the convergence of the +BP path to SLE_6, we deduce that we can decompose for sufficiently small mesh-size δ,

$$\mathbb{E}\big[\Delta_{j,n} \sum_{k=m_{n-1}+1}^{j} L_k|\mathcal{F}_{m_{n-1}}\big] \approx \mathbb{E}\big[\Delta_{j,n}\big]\mathbb{E}\big[\sum_{k=m_{n-1}+1}^{j} L_k|\mathcal{F}_{m_{n-1}}\big].$$

Idea of the Proof: Driving term convergence of the +CBP

From the definition of (L_k), using a symmetry argument, one should be able to show that

$$\mathbb{E}\Big[\sum_{k=m_{n-1}+1}^{j} L_k|\mathcal{F}_{m_{n-1}}\Big] \approx 0.$$

(at least sufficiently far from the boundary). This would imply that

$$\mathbb{E}\left[M_n - M_{n-1}|\mathcal{F}_{m_{n-1}}\right] \approx \mathbb{E}\left[R_{n-1}(t_{m_n}) - R_{n-1}(t_{m_{n-1}})|\mathcal{F}_{m_n-1}\right].$$

Hence

$$M_n - \sum_{k=1}^{n} R_{k-1}(t_{m_k}) - R_{k-1}(t_{m_{k-1}})$$

is 'almost' a martingale.

By telescoping the sum in the definition of $R_n(t)$, we can show that

$$\begin{aligned} & \Big|\sum_{k=1}^{n} R_{k-1}(t_{m_k}) - R_{k-1}(t_{m_{k-1}})\Big| \\ \leq \;& \mathcal{W}^{\delta}|A(t_{m_k}) - A(t_{m_k-1})| + |B(t_{m_k}) - B(t_{m_k-1})| \end{aligned}$$

where A and B are finite variation processes and

$$\mathcal{W}^{\delta} = \max_{j=2,\ldots m_{n-1}} \Big|\sum_{k=2}^{j} L_k\Big|.$$

$\mathcal{W}^{\delta}$ is the maximum winding of the path. By independence of unvisited disjoint rectangles, we can show that the tail probability of this $\mathcal{W}^{\delta}$ has exponential decay, and hence in particular its moments are all bounded.

Idea of the Proof: Driving term convergence of the +CBP

Then we can use a version of the Kolmogorov-Centsov continuity theorem to show that

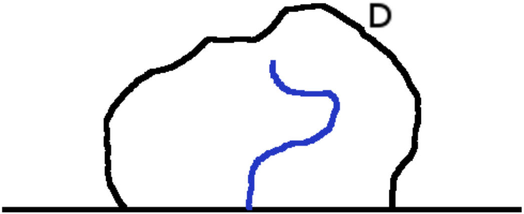
$$\sum_{k=1}^{n} R_{k-1}(t_{m_k}) - R_{k-1}(t_{m_{k-1}})$$

is a finite $(1+\epsilon)$-variation process for all sufficiently small $\epsilon > 0$, see Young (1936) and Lyons and Qian (2002).

Hence we should be able to embed M_n into a continuous time ϵ-semimartingale M_t so that ξ_t should converge (subsequentially) to M_t as the mesh size $\delta \searrow 0$.

From this we deduce that the Loewner driving function of the bond percolation exploration path also converges subsequentially to an ϵ-semimartingale.

The bond percolation exploration path satisfies the *locality property*: This means for any domain D with $0 \in \partial D$ and $D \cap \mathbb{H} \neq 0$, the bond percolation exploration process from 0 to b in D is identically distributed to the bond percolation exploration process from 0 to ∞ in $\mathbb{H}$ until first exit time of $D \cap \mathbb{H}$.



Hence conditioned on $\gamma[0,s]$, the bond percolation exploration process in $\mathbb{H}\setminus\gamma[0,s]$ is identically distributed to the bond percolation exploration process in $\mathbb{H}$ until first exit time of the common domain.

This means that the driving function of the scaling limit of the bond percolation exploration path, $W_t = \int_0^t X_s dB_s + Y_t$, satisfies a self-similarity property which leads to the following formula.

$$W_{t+s} - W_s \sim \int_0^t X_s d\widetilde{B}_s + \int_0^t \Phi_s'(W_s) dY_s + \int_0^t \left(\frac{X_s^2}{2} - 3\right) \frac{\Phi_s''(W_s)}{\Phi_s'(W_s)^2} ds$$

$$W_{t+s} - W_s \sim \int_0^t X_s d\widetilde{B}_s + \int_0^t \Phi'_s(W_s) dY_s + \int_0^t \left(\frac{X_s^2}{2} - 3\right) \frac{\Phi''_s(W_s)}{\Phi'_s(W_s)^2} ds$$

This implies that the martingale part of W_t is infinitely divisible. The Lévy-Khintchine Theorem implies that the martingale part must be $\sqrt{\kappa} B_t$ for some $\kappa \in \mathbb{R}$.

Similarly, we can show that Y_t is of finite variation using infinite divisibility. A Girsanov's Theorem argument then implies that $\kappa = 6$. Then symmetry and infinite divisibility again imply that $Y_t = 0$. Hence $W_t = \sqrt{6} B_t$.

Since every subsequential limit is $\sqrt{6} B_t$ this implies that the driving function of the full scaling limit must be $\sqrt{6} B_t$.

We either use the machinery of Camia and Newman (2007) or the recent result of Sheffield and Sun (2012) to deduce that the bond percolation exploration path converges to SLE_6 in the scaling limit. Argument of Sheffield and Sun: seeing the bond percolation exploration path from a point other infinity, and hence a radial Loewner differential equations results:

$$\dot{g}_t(z) = g_t(z)\frac{e^{i\lambda_t} + g_t(z)}{e^{i\lambda_t} - g_t(z)},$$

where $g_t(\gamma(t)) = e^{i\lambda_t}$ is the radial driving function. Using essentially the same argument for chordal version, same conclusion results.

This proves the whole theorem.

The End.