A generalization of Jarník-Besicovitch theorem by continued fraction

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Gauss transformation, Partial quotient

Gauss transformation $T : [0,1) \to [0,1)$:

$$T(0) := 0, \quad T(x) := \frac{1}{x} (\text{mod } 1), \text{ for } x \in (0,1).$$

$x \in (0,1) \cap \mathbb{Q}^c$:

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}} = [a_1(x), a_2(x), a_3(x), \cdots],$$

Partial quotient of x: $a_n(x) = \left[\frac{1}{T^{n-1}x}\right] = a_1(T^{n-1}(x))$ $(n \geq 1)$.

Convergent, cylinder

Convergent of order n of x:

$$\frac{p_n(x)}{q_n(x)} = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \ddots + \cfrac{1}{a_n(x)}}} = [a_1(x), a_2(x), \cdots, a_n(x)],$$

$$p_n(x) = a_n(x) \cdot p_{n-1}(x) + p_{n-2}(x),$$

$$q_n(x) = a_n(x) \cdot q_{n-1}(x) + q_{n-2}(x).$$

Cylinder of order n:

$$I_n(a_1, \cdots, a_n) = \{x \in [0,1) \colon a_k(x) = a_k, 1 \le k \le n\}.$$

Best approximation

Theorem (Lagrange)

Every convergent of x is a best approximation of x, i.e., if $\frac{p}{q} \neq \frac{p_n(x)}{q_n(x)}, 0 < q \leq q_n(x)$ there follows

$$|x - \frac{p}{q}| > |x - \frac{p_n(x)}{q_n(x)}| \quad \text{and} \quad |qx - p| > |q_n(x)x - p_n(x)|.$$

Theorem (Legendre)

Every irreducible fraction $\frac{p}{q}$ satisfying the inequality

$$|x - \frac{p}{q}| < \frac{1}{2q^2}$$

is a convergent of x.

Upper and lower bounds

Upper and lower bounds for Diophantine approximations:

$$\frac{1}{(a_{n+1}(x)+2)q_n^2(x)} \leq \left|x - \frac{p_n(x)}{q_n(x)}\right| \leq \frac{1}{a_{n+1}(x)q_n^2(x)}$$

A badly approximable number is an x for which there exists $c > 0$ such that for all rational $\frac{p}{q}$, we have

$$\left|x - \frac{p}{q}\right| > \frac{c}{q^2}.$$

The badly approximable numbers are precisely those with bounded partial quotients.

Well approximable sets

A number x is said to be τ-well approximable if one can find infinitely many p/q such that $|x - p/q| < q^{-\tau}$.
Jarník-Besicovitch set:

$$J(\tau) := \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau}, \text{ for infinitely many } \frac{p}{q} \right\}.$$

Irrationality exponent of x: $\delta(x) = \sup\{\tau : x \in J(\tau)\}$.
For any irrational $x \in [0,1]$, $\delta(x) \geq 2$.

Theorem (Jarník, 1931 & Besicovitch, 1934)

For any $\tau \geq 2$,

$$\dim_{\mathrm{H}} \{x \in [0,1] \colon \delta(x) \geq \tau\} = \dim_{\mathrm{H}} \{x \in [0,1] \colon \delta(x) = \tau\} = \frac{2}{\tau}.$$

General error function

For a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, denote

$$W(\psi) = \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < \psi(q), \text{ for infinitely many } \frac{p}{q} \right\}$$

and

$$Exact(\psi) = W(\psi) \setminus \bigcup_{k \geq 2} W((1 - \frac{1}{k})\psi).$$

Theorem (Bugeaud,2003)

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that $x \mapsto x^2\psi(x)$ is non-increasing and $\sum_{x \geq 1} x\psi(x) < \infty$, then

$$\dim_{\mathrm{H}} Exact(\psi) = \dim_{\mathrm{H}} W(\psi) = \frac{2}{\lambda},$$

where $\lambda = \liminf_{x \to \infty} \frac{-\log \psi(x)}{\log x}$.

A localized Jarník-Besicovitch set

Theorem (Barral and Seuret, 2011)

Let $f : [0,1] \to [2,+\infty)$ *be a continuous function,*

$$\dim_{\mathrm{H}}\{x \in [0,1] \colon \delta(x) \geq f(x)\} = \dim_{\mathrm{H}}\{x \in [0,1] \colon \delta(x) = f(x)\}$$
$$= \frac{2}{\min\{f(x) : x \in [0,1]\}}.$$

Applications:(Barral and Seuret)

- For all real numbers $0 < a < b < 1$,
$$\dim_{\mathrm{H}} \{x \in [a,b] \colon \delta(x) = 2(1+x)\} = \frac{1}{a+1};$$
- for all real numbers $\frac{1}{6} < a < b < \frac{5}{6}$,
$$\dim_{\mathrm{H}} \{x \in [a,b] \colon \delta(x) = 4\sin(\pi x)\} = \frac{1}{2\min\{\sin(\pi a), \sin(\pi b)\}}.$$

1. Continued fraction
2. Diophantine approximation
3. A generalized Jarník-Besicovitch set

Jarník-Besicovitch revisited

Jarník-Besicovitch set can be revisited, in terms of continued fractions, as

$$\left\{x\colon a_{n+1}(x) \geq q_n^{\tau-2}(x), \quad \text{i.o. } n \in \mathbb{N}\right\},$$

or

$$\left\{x\colon a_{n+1}(x) \geq e^{\frac{\tau-2}{2}(\log|T'x|+\cdots+\log|T'(T^{n-1}x)|)}, \quad \text{i.o. } n \in \mathbb{N}\right\},$$

by noticing that $q_n^2(x) \leq \left|(T^n)'(x)\right| \leq 4q_n^2(x)$.

A generalized Jarník-Besicovitch set

Let f, g be two positive functions defined on $[0,1]$, denote by $S_n g(x)$ for the ergodic sum $g(x) + \cdots + g(T^{n-1}x)$. Define

$$D(f,g) = \Big\{x \in [0,1] \colon a_{n+1}(x) \geq e^{f(x)S_n g(x)}, \text{ i.o. } n \in \mathbb{N}\Big\}$$

and

$$E(f,g) = \bigcap_{\varepsilon>0} \Big\{x \in [0,1] \colon a_{n+1}(x) \geq e^{(1-\varepsilon)f(x)S_n g(x)}, \text{ i.o.}, \text{ but } a_{n+1}(x) \leq e^{(1+\varepsilon)f(x)S_n g(x)}, \text{ eventually}\Big\}.$$

For special case when $g(x) = \log|T'x|$,

- $D(f,g) = \{x \in [0,1] \colon \delta(x) \geq 2(f(x)+1)\}$,
- $E(f,g) = \{x \in [0,1] \colon \delta(x) = 2(f(x)+1)\}$.

Notations

Tempered Distortion Property: if ϕ satisfying

$$\mathrm{Var}_1(\phi) < \infty \text{ and } \lim_{n\to\infty} \frac{\mathrm{Var}_n(S_n(\phi))}{n} = 0,$$

where $\mathrm{Var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : I_n(x) = I_n(y)\}$ denotes the n-th variation of ϕ.
Let $A \subset \mathbb{N}$, define

$$X_A = \{x \in [0,1] : a_n(x) \in A, \text{for all } n \geq 1\}.$$

Let ϕ be a potential function satisfying the tempered distortion property, the pressure function restricted to the subsystem (X_A, T) is defined by

$$P_A(T,\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,\cdots,a_n)\in A^n} e^{-S_n\phi([a_1,\cdots,a_n])}.$$

When $A = \mathbb{N}$, also denote $P_{\mathbb{N}}(T,\phi)$ by $P(T,\phi)$.

Main results

Theorem

Let $f : [0,1] \to \mathbb{R}^+$ *be a continuous function and let* $g : [0,1] \to \mathbb{R}^+$ *satisfies the tempered distortion condition. Then*

$$\dim_{\mathrm{H}} D(f,g) = \inf \left\{ s \colon P(T, s(f_* g + \log |T'|)) \le 0 \right\},$$

where $f_* = \min\{f(x) \colon x \in [0,1]\}$.

Theorem

Let $f : [0,1] \to \mathbb{R}^+$ *be a continuous function and let* $g : [0,1] \to \mathbb{R}^+$ *satisfies:*

- *the tempered distortion condition;*
- *a growth condition:* $\forall x \in [0,1],\ \liminf_{n\to\infty} \frac{S_n g(x)}{n} \ge \delta > 0.$

Then $\dim_{\mathrm{H}} E(f,g) = \inf \left\{ s \colon P(T, s(f_* g + \log |T'|)) \le 0 \right\}.$

Outline of the proof of the second result

Let $A \subset \mathbb{N}$. For each $n \geq 1$ and $s \geq 0$, let

$$\mathcal{N}_{n,A}(s) = \sum_{a_1,\cdots,a_n \in A} \left(\frac{1}{e^{f_* S_n g(y)} q_n^2(y)} \right)^s,$$

where $y = [a_1, \cdots, a_n]$. Define

$$s_n(A) = \inf\{s \geq 0 : \mathcal{N}_{n,A}(s) \leq 1\},$$
$$s(A) = \inf\{s : P_A(T, s(f_* g + \log |T'|)) \leq 0\}.$$

When $A = \{1, \cdots, \beta\}$, write $s(\beta)$ for $s(A)$.

Lemma

$$\lim_{n\to\infty} s_n(\beta) = s(\beta), \ \lim_{n\to\infty} s_n(\mathbb{N}) = s(\mathbb{N}), \ \lim_{\beta\to\infty} s(\beta) = s(\mathbb{N}).$$

Upper bound: Using natural cover.
Lower bound: Constructing Cantor set. For a fixed $\varepsilon > 0$. Let $y_0 \in [0,1]$ be an irrational and $t_0 \in \mathbb{N}$ such that for any $n \geq t_0$, $f(y) \leq (1+\varepsilon) f_*$ for all $y \in I_n(y_0)$.
Level 1'. $(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)})$:

$$1 \leq b_1^{(1)}, \cdots, b_{m_1}^{(1)} \leq \beta.$$

Level 1. $(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1})$:

$$e^{f(y_1) S_{l_1} g(y_1)} \leq a_{n_1} < 2 e^{f(y_1) S_{l_1} g(y_1)},$$

where $l_1 = n_1 - 1$ and $y_1 = [a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}]$.
Level 2'. For each $(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1}) \in Level\ 1$,

$$(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1},$$
$$a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_1^{(2)}, \cdots, b_{m_2}^{(2)}) :$$
$$1 \leq b_1^{(2)}, \cdots, b_{m_2}^{(2)} \leq \beta.$$

Level 2.

$$(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1},$$
$$a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_1^{(2)}, \cdots, b_{m_2}^{(2)}, a_{n_2}) :$$
$$e^{f(y_2)S_{l_2}g(y_2)} \le a_{n_2} < 2e^{f(y_2)S_{l_2}g(y_2)},$$

where $l_2 = n_2 - n_1 - 1$ and $y_2 = [a_1(y_0), \cdots, b_1^{(1)}, b_1^{(2)}, \cdots, b_{m_2}^{(2)}]$.

Level 3'. For each $(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_1^{(2)}, \cdots, a_{n_2}) \in Level\ 2$,

$$(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_2^{(1)}, \cdots, b_{m_1}^{(1)}, a_{n_1},$$
$$a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_1^{(2)}, \cdots, b_{m_2}^{(2)}, a_{n_2},$$
$$a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, b_2^{(1)}, b_1^{(3)}, \cdots, b_{m_3}^{(3)}) :$$
$$1 \le b_1^{(3)}, \cdots, b_{m_3}^{(3)} \le \beta.$$

The next level is constructed by looking locally at each block from the previous level.

Notice that $\lim\limits_{k\to\infty} \frac{f(y_k)S_{l_k}g(y_k)}{f(x)S_{n_k-1}g(x)} = 1.$

Supporting measure

Mass distribution of *Level 1*:

$$\mu\Big(I_{t_0}(a_1(y_0), \cdots, a_{t_0}(y_0))\Big) = 1.$$

Define s_1 as the solution to

$$\sum_{a_1=a_1(y_0),\cdots,a_{t_0}=a_{t_0}(y_0),\ 1\le b_1^{(1)},\cdots,b_{m_1}^{(1)}\le\beta} \left(\frac{1}{e^{f(y)S_{l_1}g(y)}q_{l_1}^2(y)}\right)^s = 1,$$

where $y = [a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)}]$.
Then for each $I_{l_1}(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)})$, define

$$\mu\Big(I_{l_1}(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{m_1}^{(1)})\Big) = \left(\frac{1}{e^{f(y)S_{l_1}g(y)}q_{l_1}^2(y)}\right)^{s_1}.$$

When $n > t_1$, for the predecessors $I_n(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{n-t_0}^{(1)})$ of $I_{l_1} \in Level\ 1$, define

$$\mu\Big(I_n(a_1(y_0), \cdots, a_{t_0}(y_0), b_1^{(1)}, \cdots, b_{n-t_0}^{(1)})\Big)$$
$$= \sum_{1 \le b_{n-t_0+1}^{(1)}, \cdots, b_{m_1}^{(1)} \le \beta} \mu\Big(I_{l_1}(a_1(y_0), \cdots, b_{n-t_0}^{(1)}, b_{n-t_0+1}^{(1)}, \cdots, b_{m_1}^{(1)})\Big).$$

For each $I_{n_1}(a_1(y_0), \cdots, b_{m_1}^{(1)}, a_{n_1}) \in Level\ 1$, define

$$\mu\Big(I_{n_1}(a_1(y_0), \cdots, b_{m_1}^{(1)}, a_{n_1})\Big) = \frac{1}{e^{f(y_1)S_{l_1}g(y_1)}}\mu\Big(I_{l_1}(a_1(y_0), \cdots, b_{m_1}^{(1)})\Big),$$

where $y_1 = [a_1(y_0), \cdots, b_{m_1}^{(1)}]$, i.e., the measure of I_{l_1} are evenly distributed on its offsprings.
Notice that $|s_1 - s(\mathbb{N})| < 3\varepsilon$.

Remark

Remark

In the second result, g is assumed to fulfill a growth condition, we remark that if g satisfying the bounded distortion property, then the result is still valid without the growth assumption of g.

Application: $f(x) = \tau$, $g(x) = \log a_1(x)$,

$$\dim_{\mathrm{H}} \left\{ x \in [0,1] \colon a_{n+1}(x) \geq \prod_{i=1}^{n} a_i^{\tau}(x), \text{ i.o. } n \right\}$$

$$= \dim_{\mathrm{H}} \bigcap_{\varepsilon > 0} \left\{ x \in [0,1] \colon a_{n+1}(x) \geq \prod_{i=1}^{n} a_i^{(1-\varepsilon)\tau}(x), \text{ i.o.}, \right.$$

$$\left. \text{but } a_{n+1}(x) \leq \prod_{i=1}^{n} a_i^{(1+\varepsilon)\tau}(x), \text{ eventually} \right\}$$

$$= \inf \left\{ s \colon P(T, s(\tau \log a_1 + \log |T'|)) \leq 0 \right\}.$$

Thanks

THANK YOU!