Ideal Class and Lipschitz Equivalent Class

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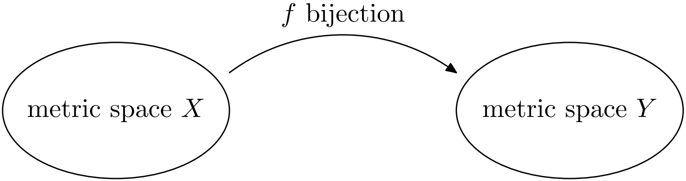
Lipschitz equivalence ($X \simeq Y$)



$$C^{-1} \leq \frac{d_Y\big(f(x_1), f(x_2)\big)}{d_X(x_1, x_2)} \leq C$$

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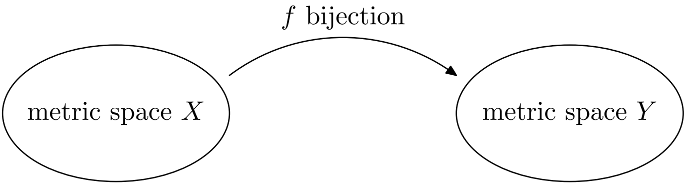

$$C^{-1} \leq \frac{d_Y\big(f(x_1), f(x_2)\big)}{d_X(x_1, x_2)} \leq C$$

$\dim_H X = \dim_H Y$: same size

$X \simeq Y$: same geometric structure

- This talk concerns the Lipschitz equivalence of self-similar sets.
- Our result establishes a one-to-one correspondence between the Lipschitz equivalence classes of self-similar sets and the ideal classes in a related ring.
- This reveals an interesting relationship between the Lipschitz class number problem and the Gauss class number problems.

Notations (I)

The ratios $r_1, \ldots, r_N$ of IFS $\mathcal{S}$ are commensurable if

$$\log r_i / \log r_j \in \mathbb{Q} \quad \text{for } 1 \leq i, j \leq N.$$

In this case, $\exists! r_{\mathcal{S}} \in (0,1)$ such that

$$\mathrm{mgp}(r_1, \ldots, r_N) = \mathrm{mgp}(r_{\mathcal{S}}).$$

Write $p_{\mathcal{S}} = r_{\mathcal{S}}^s$, ($s = \dim_{\mathrm{H}} E_{\mathcal{S}}$).

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Write $p_{\mathcal{S}} = r_{\mathcal{S}}^s$, ($s = \dim_{\mathrm{H}} E_{\mathcal{S}}$).

Example

IFS $\mathcal{S}$ with ratios $\underbrace{r, \ldots, r}_{N}$.

Then $p_{\mathcal{S}} = 1/N$ be the positive solution of $\underbrace{p + \cdots + p}_{N} = 1$

and $r_{\mathcal{S}} = r$.

IFS $\mathcal{T}$ with ratios r^3, r^2, r^2.

Then $p_{\mathcal{T}} = (\sqrt{5} - 1)/2$ be the positive solution of $p^3 + p^2 + p^2 = 1$ and $r_{\mathcal{T}} = r$.

Notations (II)

$$\begin{aligned}
\mathrm{TDC} &= \{\mathcal{S}\colon E_{\mathcal{S}} \text{ is totally disconnected}\},\\
\mathrm{OSC}_1 &= \{\mathcal{S}\colon E_{\mathcal{S}} \subset \mathbb{R}^d, \text{ OSC holds, ratios are commensurable}\},\\
\mathrm{OSC}_1(p,r) &= \{\mathcal{S} \in \mathrm{OSC}_1\colon p_{\mathcal{S}} = p, r_{\mathcal{S}} = r\}.
\end{aligned}$$

Ideal class and class number

$I \subset R(+, \cdot)$ is a ideal if

(i) $(I, +)$ is a group; (ii) $a \cdot I \subset I$ for all $a \in R$.

Example (Principle ideal)

For any $a \in R$, the set $a \cdot R$ is an ideal of R. Such ideal is called a principle ideal, denoted by (a).

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I, J: two ideals of R.

- $I \sim J$: $aI = bJ$ for some $a, b \in R$;
  e.g., if $I = (a_0)$, $J = (b_0)$, then $b_0 I = a_0 J = (a_0 b_0)$.

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- Ideal classes: the corresponding equivalence classes.
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Example

The class number of $\mathbb{Z}[\sqrt{10}]$ is 2.

In fact, ideal $(2, \sqrt{10}) \subset \mathbb{Z}[\sqrt{10}]$ is not a principle ideal.

Ideal class and Lipschitz equivalent class

Theorem (Suppose $\mathrm{TDC} \cap \mathrm{OSC}_1(p, r) \neq \emptyset$)

The Lipschitz equivalent classes of self-similar sets generated by IFS in $\mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ correspond one-to-one to the ideal classes of $\mathbb{Z}[p]$.

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This theorem means that, each $\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ corresponds to an ideal class $\mathcal{I}_\mathcal{S}$ of $\mathbb{Z}[p]$ such that

1. $E_\mathcal{S} \simeq E_\mathcal{T} \Longleftrightarrow \mathcal{I}_\mathcal{S} = \mathcal{I}_\mathcal{T}$.
2. For any ideal class $\mathcal{I}$ of $\mathbb{Z}[p]$, there exists an $\mathcal{S} \in \mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ with $\mathcal{I}_\mathcal{S} = \mathcal{I}$.

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Theorem

The Lipschitz class number of $\mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ is finite.

Gauss class number problem

Lipschitz class number problem

Given $n > 0$, determine all p, r such that the Lipschitz class number of $\mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ is n.

The Lipschitz class number problem is closely related to the Gauss class number problems. For example,

Gauss class number one problem for real quadratic fields

There are infinitely many square free $D > 0$ such that the class number of $\mathcal{O}_D$ is one, where $\mathcal{O}_D$ denotes the ring of all the algebraic integers of $\mathbb{Q}(\sqrt{D})$.

This conjecture was proposed by Gauss in 1801 but still remains a open question today.

Lipschitz class number one

$\mathbb{Z}[p]$ is a principle ideal domain $\Longleftrightarrow$ $\mathbb{Z}[p]$ with class number one

$\Longleftrightarrow \mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ with Lipschitz class number one

$\mathbb{Z}[p]$ is a principle ideal domain when $p = 1/N, \sqrt{2} - 1, (\sqrt{3} - 1)/2, \ldots$.

Lipschitz class number one

$\mathbb{Z}[p]$ is a principle ideal domain $\iff \mathbb{Z}[p]$ with class number one
$\iff \mathrm{TDC} \cap \mathrm{OSC}_1(p, r)$ with Lipschitz class number one

$\mathbb{Z}[p]$ is a principle ideal domain when $p = 1/N, \sqrt{2} - 1, (\sqrt{3} - 1)/2, \ldots$.

Theorem

Suppose that $\mathcal{S} = \{S_1, \ldots, S_N\}$, $\mathcal{T} = \{T_1, \ldots, T_N\}$ and

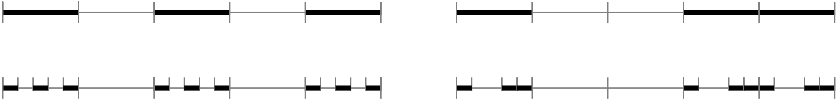
- *$\mathcal{S}, \mathcal{T}$ satisfy the OSC;*
- *all the ratios of S_i and T_j equal to r;*
- *$E_{\mathcal{S}}, E_{\mathcal{T}} \subset \mathbb{R}^d$ are totally disconnected.*

Then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$.

Proof.

$\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1(1/N, r)$ and $\mathbb{Z}[1/N]$ is a principle ideal domain. □

Example: {1,3,5}-{1,4,5} problem



$\{1,3,5\}$-$\{1,4,5\}$ problem, by David & Semmes

Rao, Ruan & Xi, 2006 $E_{1,3,5} \simeq E_{1,4,5}$, graph-directed system.

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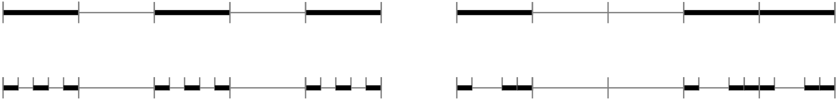

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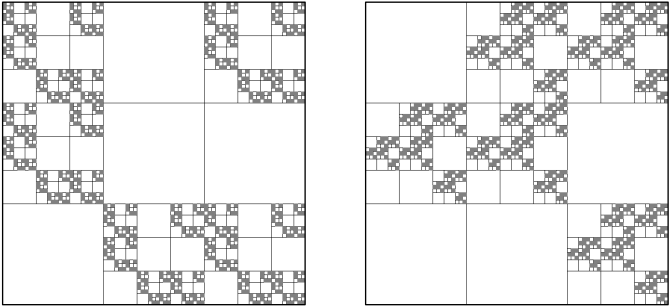


Xi & Xiong, 2010 higher dimensional Euclidean spaces.

Example: $\mathbb{Z}[(\sqrt{5}-1)/2]$ has class number one

Example

IFS $\mathcal{S}$ OSC & TDC, with ratios r^4, r^3, r

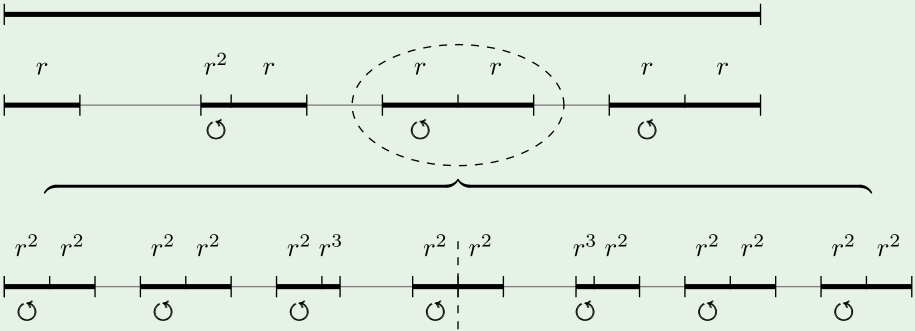
IFS $\mathcal{T}$ OSC & TDC, with ratios r^3, r^2, r^2

then $p_{\mathcal{S}} = p_{\mathcal{T}} = (\sqrt{5}-1)/2$ and $r_{\mathcal{S}} = r_{\mathcal{T}} = r$. Since $\mathbb{Z}[(\sqrt{5}-1)/2]$ has class number one, we have $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$.

In this example, the relative positions of the small copies of self-similar sets $E_{\mathcal{S}}$ and $E_{\mathcal{T}}$ does not affect the Lipschitz equivalence.

Example: $\mathbb{Z}[\sqrt{10}]$ has class number two

Example



IFS $\mathcal{S}$ in above figure, with ratios r, r, r, r, r, r, r^2

IFS $\mathcal{T}$ SSC, with ratios r, r, r, r, r, r, r^2

Then $\mathcal{S}, \mathcal{T} \in \mathrm{TDC} \cap \mathrm{OSC}_1(\sqrt{10} - 3, r)$ and $E_{\mathcal{S}} \not\simeq E_{\mathcal{T}}$.

SSC corresponds to the principle ideal class

Theorem

Suppose that $\mathcal{S}, \mathcal{T}$ both satisfy the SSC and the ratios of them are both commensurable. Then $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$ if and only if

1. $\dim_{\mathrm{H}} E_{\mathcal{S}} = \dim_{\mathrm{H}} E_{\mathcal{T}}$;
2. $\log r_{\mathcal{S}} / \log r_{\mathcal{T}} \in \mathbb{Q}$;
3. $\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[p_{\mathcal{T}}]$.

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Necessary conditions in non-commensurable case (Falconer & Marsh)

Suppose that $\mathcal{S}, \mathcal{T}$ both satisfy the SSC and $r_1, \ldots, r_n$ are ratios of $\mathcal{S}$, $t_1, \ldots, t_m$ are ratios of $\mathcal{T}$. If $E_\mathcal{S} \simeq E_\mathcal{T}$, then

1. $\dim_{\mathrm{H}} E_\mathcal{S} = \dim_{\mathrm{H}} E_\mathcal{T} = s$;
2. $\mathrm{sgp}(r_1^u, \ldots, r_n^u) \subset \mathrm{sgp}(t_1, \ldots, t_m)$, $\mathrm{sgp}(t_1^v, \ldots, t_m^v) \subset \mathrm{sgp}(r_1, \ldots, r_n)$.
3. $\mathbb{Q}(r_1^s, \ldots, r_n^s) = \mathbb{Q}(t_1^s, \ldots, t_m^s)$;

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Theorem

If $E_{\mathcal{S}} \simeq E_{\mathcal{T}}$, then $\mathbb{Z}[r_1^s, \ldots, r_n^s] = \mathbb{Z}[t_1^s, \ldots, t_m^s]$.

Example: SSC

Example (the ring condition does stronger than the field condition)

$$p_{\mathcal{S}} = \frac{\sqrt{3}-1}{2} : \text{the positive solution of } 2p_{\mathcal{S}}^2 + 2p_{\mathcal{S}} = 1.$$

$$p_{\mathcal{T}} = \frac{3\sqrt{3}-5}{4} : \text{the positive solution of } 8p_{\mathcal{T}}^2 + 20p_{\mathcal{T}} = 1.$$

Then

$$\log p_{\mathcal{S}} / \log p_{\mathcal{T}} = \frac{1}{3} \in \mathbb{Q}, \quad \mathbb{Q}(p_{\mathcal{S}}) = \mathbb{Q}(p_{\mathcal{T}}) = \mathbb{Q}(\sqrt{3}),$$

but

$$\mathbb{Z}[p_{\mathcal{S}}] = \mathbb{Z}[\sqrt{3}, \frac{1}{2}] \neq \mathbb{Z}[p_{\mathcal{T}}] = \mathbb{Z}[3\sqrt{3}, \frac{1}{2}].$$

Example: non-commensurable

Example (these necessary conditions are far from being sufficient)

Ratios of E_1 $1/9$ and $4/9$

Ratios of E_2 $1/81, 1/81, 1/81, 1/81$ and $4/9$

$\cdots$ $\cdots$

Ratios of E_n $\underbrace{9^{-n}, \ldots, 9^{-n}}_{3^{n-1}}$ and $4/9$

$\cdots$ $\cdots$

Then

1. $\dim_{\mathrm{H}} E_n = 1/2$ for all $n \geq 1$
2. $\mathrm{sgp}((9^{-n})^m, (4/9)^m) \subset \mathrm{sgp}(9^{-m}, 4/9)$ and $\mathrm{sgp}((9^{-m})^n, (4/9)^n) \subset \mathrm{sgp}(9^{-n}, 4/9)$ for all m, n
3. $\mathbb{Z}[3^{-n}, 2/3] = \mathbb{Z}[1/3]$ for all $n \geq 1$

But $E_m \not\simeq E_n$ for $m \neq n$.

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Thank you!