Brownian Motion and Thermal Capacity

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Outline

- Intersection of the Brownian images and thermal capacity
- Hausdorff dimension of $W(E) \cap F$
- Further research and open problems

1. Intersection of the Brownian images and thermal capacity

Let $W := \{W(t)\}_{t \geq 0}$ denote standard d-dimensional Brownian motion where $d \geq 1$, and let E and F be compact subsets of $(0, \infty)$ and $\mathbb{R}^d$, respectively.

The following problems are of interest:

1. When is $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$?
2. What is $\dim_{\mathrm{H}}(W(E) \cap F)$?

Note that

$$\{W(E) \cap F \neq \emptyset\} = \{(t, W(t)) \in E \times F \text{ for some } \ t > 0\}.$$

Problem 1 is an interesting problem in probabilistic potential theory.

Conditions for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$

Necessary and sufficient condition in terms of "thermal capacity" for $\mathbb{P}(W(E) \cap F \neq \emptyset) > 0$ were proved by Waston (1978) and Doob (1984).

Waston and Taylor (1985) provided a simple-to-use condition:

$$\mathbb{P}(W(E) \cap F \neq \emptyset) \begin{cases} > 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) > d, \\ = 0, & \text{if } \dim_{\mathrm{H}}(E \times F\,; \varrho) < d. \end{cases}$$

In the above, $\dim_{\mathrm{H}}(E \times F\,; \varrho)$ is the Hausdorff dimension of $E \times F$ using the metric

$$\varrho\left((s\,,x)\,;(t\,,y)\right) := \max\left(|t-s|^{1/2},\ \|x-y\|\right).$$

As a by-product of our main result, we obtain a slightly improved version of the result of Waston (1978) and Doob (1984).

Theorem 1.1

Suppose $F \subset \mathbb{R}^d$ ($d \geq 1$) is compact and has Lebesgue measure 0. Then

$$\mathbb{P}\{W(E) \cap F \neq \emptyset\} > 0 \iff \\ \exists\, \mu \in \mathcal{P}_d(E \times F) \text{ such that } \mathcal{E}_0(\mu) < \infty,$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and the energy $\mathcal{E}_0(\mu)$ will be defined below.

2. Hausdorff dimension of $\dim_{\mathrm{H}}(W(E) \cap F)$

Two common ways to compute the Hausdorff dimension of a set:

- Use a covering argument for obtaining an upper bound and a capacity argument for lower bound;
- The co-dimension argument.

The co-dimension argument

The "co-dimension argument" was initiated by S.J. Taylor (1966) for computing the Hausdorff dimension of the multiple points of a stable Lévy process in $\mathbb{R}^d$. His method was based on potential theory of Lévy processes.

Let $Z_\alpha = \{Z_\alpha(t), t \in \mathbb{R}_+\}$ be a (symmetric) stable Lévy process in $\mathbb{R}^d$ of index $\alpha \in (0, 2]$ and let $F \subset \mathbb{R}^d$ be a Borel set. Then

$$\mathbb{P}(Z_\alpha((0, \infty)) \cap F \neq \varnothing) > 0 \iff \mathrm{Cap}_{d-\alpha}(F) > 0,$$

where $\mathrm{Cap}_{d-\alpha}$ is the Riesz-Bessel capacity of order $d - \alpha$.

The co-dimension argument

The above result and Frostman's theorem lead to the *stochastic co-dimension argument*: If $\dim_{\rm H} F \ge d-2$, then

$$\begin{aligned}\dim_{\rm H} F &= \sup\{d-\alpha : Z_\alpha((0,\infty)) \cap F \neq \varnothing\} \\ &= d - \inf\big\{\alpha > 0 : \ F \ \text{is not polar for } Z_\alpha\big\}.\end{aligned}$$

[The restriction $\dim_{\rm H} F \ge d-2$ is caused by the fact that $Z_\alpha((0,\infty)) \cap F = \emptyset$ if $\dim_{\rm H} F < d-2$.]

This method determines $\dim_{\rm H} F$ by intersecting F using a family of testing random sets.

Hawkes (1971) applied the co-dimension method for computing the Hausdorff dimension of the inverse image $X^{-1}(F)$ of a stable Lévy process.

The co-dimension argument

Families of testing random sets:

- ranges of symmetric stable Lévy processes;
- fractal percolation sets [Peres (1996, 1999)];
- ranges of additive Lévy processes [Khoshnevisan and X. (2003, 2005), Khoshnevisan, Shieh and X. (2008)].

Hausdorff dimension of $\dim_{\mathrm{H}}(W(E) \cap F)$

If $F = \mathbb{R}^d$, then $\dim_{\mathrm{H}} W(E) = \min\{d, 2\dim_{\mathrm{H}} E\}$ a.s.

In general, $\dim_{\mathrm{H}}(W(E) \cap F)$ is a (non-degenerate) random variable, an example was shown to us by Greg Lawler.

Hence we compute $\|\dim_{\mathrm{H}} (W(E) \cap F)\|_{L^\infty(\mathbb{P})}$, the $L^\infty(\mathbb{P})$-norm of $\dim_{\mathrm{H}}(W(E) \cap F)$.

We distinguish two cases: $|F| > 0$ and $|F| = 0$, where $|\cdot|$ denotes the Lebesgue measure.

Theorem 2.1 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and $|F| > 0$, then

$$\|\dim_{\mathrm{H}} (W(E) \cap F)\|_{L^\infty(\mathbb{P})} = \min\{d\,, 2\dim_{\mathrm{H}} E\}. \qquad (1)$$

If $\dim_{\mathrm{H}} E > \frac{1}{2}$ and $d = 1$, then $\mathbb{P}\{|W(E) \cap F| > 0\} > 0$.

Proof of Theorem 2.1

Thanks to the uniform Hölder continuity of $W(t)$ on bounded sets, we have

$$\dim_{\mathrm{H}} \left(W(E) \cap F\right) \le \min\{d\,, 2\dim_{\mathrm{H}} E\}, \qquad \text{a.s.}$$

This implies the upper bound in (1).

For proving the lower bound in (1), we construct a random measure on $W(E) \cap F$ and use the capacity argument.

The last part is proved by showing that the constructed random measure on $W(E) \cap F$ has a density function almost surely.

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Theorem 2.2 [Khoshnevisan and X. (2012)]

If $F \subset \mathbb{R}^d$ $(d \geq 1)$ is compact and $|F| = 0$, then

$$\begin{aligned}&\left\| \dim_{\mathrm{H}} \left(W(E) \cap F \right) \right\|_{L^\infty(\mathbb{P})} \\ &\qquad = \sup\left\{ \gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E \times F)} \mathcal{E}_\gamma(\mu) < \infty \right\},\end{aligned} \tag{2}$$

where $\mathcal{P}_d(E \times F)$ is the collection of all probability measures μ on $E \times F$ such that $\mu(\{t\} \times F) = 0$ for all $t > 0$, and

$$\mathcal{E}_\gamma(\mu) := \iint \frac{e^{-\|x-y\|^2/(2|t-s|)}}{|t-s|^{d/2} \cdot \|y-x\|^\gamma} \, \mu(ds\, dx)\, \mu(dt\, dy). \tag{3}$$

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Hitting probability of random fields

We prove Theorem 2.2 by checking whether or not $W(E) \cap F$ intersects the (closure of the) range of an additive Lévy stable process.

Let $X^{(1)}, \ldots, X^{(N)}$ be N isotropic stable processes with common stability index $\alpha \in (0, 2]$. We assume that the $X^{(j)}$'s are independent from one another, as well as from the process W, and all take their values in $\mathbb{R}^d$.

We assume also that $X^{(1)}, \ldots, X^{(N)}$ have right-continuous sample paths with left-limits and

$$\mathbb{E}\left[e^{i\langle \xi, X^{(k)}(1)\rangle}\right] = e^{-\|\xi\|^\alpha/2}, \quad \forall\ \xi \in \mathbb{R}^d.$$

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$$\mathbb{E}\left[e^{i\langle \xi, X^{(k)}(1)\rangle}\right] = e^{-\|\xi\|^\alpha/2}, \quad \forall\ \xi \in \mathbb{R}^d.$$

Define the corresponding additive stable process $X_\alpha := \{X_\alpha(\boldsymbol{t}),\, \boldsymbol{t} \in \mathbb{R}_+^N\}$ as

$$X_\alpha(\boldsymbol{t}) := \sum_{k=1}^{N} X^{(k)}(t_k), \quad \forall\, \boldsymbol{t} = (t_1, \ldots, t_N) \in \mathbb{R}_+^N. \tag{4}$$

Khoshnevisan (2002) showed that for any Borel set $G \subset \mathbb{R}^d$,

$$\mathbb{P}\Big(\overline{X_\alpha(\mathbb{R}_+^N)} \cap G \neq \emptyset\Big) \begin{cases} = 0 & \text{if } \dim_{\mathrm{H}}(G) < d - \alpha N, \\ > 0 & \text{if } \dim_{\mathrm{H}}(G) > d - \alpha N. \end{cases} \tag{5}$$

The key ingredient for proving Theorem 2.2

Theorem 2.3

If $d > \alpha N$ and $F \subset \mathbb{R}^d$ has Lebesgue measure 0, then

$$\mathbb{P}\left\{W(E) \cap \overline{X_\alpha(\mathbb{R}^N_+)} \cap F \neq \varnothing\right\} > 0$$
$$\iff \mathcal{C}_{d-\alpha N}(E \times F) > 0.$$

Here and in the sequel, $\overline{A}$ denotes the closure of A, and $\mathcal{C}_\gamma$ is the capacity corresponding to the energy form (3): for all compact sets $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $\gamma \geq 0$,

$$\mathcal{C}_\gamma(U) := \left[\inf_{\mu \in \mathcal{P}_d(U)} \mathcal{E}_\gamma(\mu)\right]^{-1}. \tag{6}$$

Proof of Theorem 2.2

Lower bound: Denote

$$\Delta := \sup\left\{\gamma \geq 0 : \inf_{\mu \in \mathcal{P}_d(E\times F)} \mathcal{E}_\gamma(\mu) < \infty\right\}. \tag{7}$$

If $\Delta > 0$ and we choose $\alpha \in (0\,,2]$ and $N \in \mathbf{Z}_+$ $0 < d - \alpha N < \Delta$. Then $\mathcal{C}_{d-\alpha N}(E \times F) > 0$. It follows from Theorem 2.3 and (5) that

$$\mathbb{P}\left\{\dim_{\rm H}\left(W(E) \cap F\right) \geq d - \alpha N\right\} > 0. \tag{8}$$

Because $d - \alpha N \in (0\,, \Delta)$ is arbitrary, we have

$$\|\dim_{\rm H}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \geq \Delta.$$

Upper bound: Similarly, Theorem 2.3 and (5) imply that

$$d - \alpha N > \Delta \ \Rightarrow \ \dim_{\rm H} \big(W(E) \cap F\big) \le d - \alpha N \quad \text{a. s.} \quad (9)$$

Hence $\|\dim_{\rm H}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \le \Delta$ whenever $\Delta \ge 0$. This proves Theorem 2.2.

Proof of Theorem 2.3: The proof of sufficiency, which is based on using a second order argument on the occupation measure, is quite standard; but the proof of the necessity is hard. We omit the details.

Upper bound: Similarly, Theorem 2.3 and (5) imply that

$$d - \alpha N > \Delta \;\Rightarrow\; \dim_{_{\rm H}} \big(W(E) \cap F\big) \le d - \alpha N \quad \text{a. s.} \quad (9)$$

Hence $\|\dim_{_{\rm H}}(W(E) \cap F)\|_{L^\infty(\mathbb{P})} \le \Delta$ whenever $\Delta \ge 0$. This proves Theorem 2.2.

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An explicit formula

Theorem 2.4 [Khoshnevisan and X. (2012)]

If $d \geq 2$ and $\dim_{\rm H}(E \times F\,;\varrho) \geq d$, then

$$\left\| \dim_{\rm H}(W(E) \cap F) \right\|_{L^\infty(\mathbb{P})} = \dim_{\rm H}(E \times F\,;\varrho) - d. \qquad (10)$$

Remarks

- Eq (10) does not always hold for $d = 1$: For $E := [0\,,1]$ and $F = \{0\}$, we have $\dim_{\rm H}(W(E) \cap F) = 0$ a.s., whereas $\dim_{\rm H}(E \times F\,;\varrho) - d = 1$.
- When $F \subset \mathbb{R}^d$ satisfies $|F| > 0$, it can be shown that

$$\dim_{\rm H}(E \times F\,;\varrho) = 2\dim_{\rm H} E + d.$$

Hence (1) coincides with (10) when $d \geq 2$.

Proof of Theorem 2.4

The proof replies on the following "uniform dimension result" of Kaufman (1968): If $\{W(t), t \in \mathbb{R}_+\}$ is a Brownian motion in $\mathbb{R}^d$ with $d \geq 2$, then

$$\mathbb{P}\big\{\dim_{_H} W(G) = 2\dim_{_H} G,\ \forall \text{ Borel sets } \ G \subset \mathbb{R}_+\big\} = 1.$$

It is sufficient to show that for all compact sets $E \subset (0, \infty)$ and $F \subset \mathbb{R}^d$,

$$\Big\| \dim_{_H} \big(E \cap W^{-1}(F)\big) \Big\|_{L^\infty(\mathbb{P})} = \frac{\dim_{_H} (E \times F\,; \varrho) - d}{2}. \tag{11}$$

When $d = 1$, the lower bound of (11) was found first by Kaufman (1972).

3. Further research and open problems

Potential theoretic results have been proved for

- the Brownian sheet: Khoshnivisan and Shi (1999), Khoshnivisan and X. (2007);
- other (more general) Gaussian random fields: X. (2009), Biermé, Lacaux and X. (2009), Chen and X. (2012);
- additive Lévy processes: Khoshnevisan and X. (2002, 2005, 2009), Khoshnevisan, Shieh and X. (2008);
- SPDEs: Dalang and Nualart (2004), Dalang, et al (2007, 2009), Dalang and Sanz Solé (2010).

However, Problems 1 and 2 have not been solved for any of them.

Thank you