The dimensional theory of continued fractions

Jun Wu

Huazhong University of Science and Technology

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Notation

- Continued fraction : $x \in [0,1)$,

$$\begin{aligned} x &= \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots}}} \\ &= [a_1(x), a_2(x), a_3(x), \cdots]. \end{aligned}$$

- Gauss Transformation : $T : [0,1) \to [0,1)$ given by

$$T(0) := 0, \quad T(x) := \frac{1}{x} - \left[\frac{1}{x}\right] \quad \text{for } x \in (0,1),$$

is called the Gauss transformation.

- Partial Quotients : For all $n \in \mathbb{N}$, we have

$$a_1(x) = \left[\frac{1}{x}\right], \quad a_n(x) = \left[\frac{1}{T^{n-1}x}\right].$$

$a_n(x)$ $(n \geq 1)$ are called the partial quotients of x.

- Convergents : Let

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), a_2(x), \cdots, a_n(x)],\ n \geq 1$$

denote the convergents of x.

- n-order Basic Intervals : For any $n \geq 1$ and $(a_1, a_2, \cdots, a_n) \in \mathbb{N}^n$, let $I(a_1, a_2, \cdots, a_n)$ be the set of numbers in $[0, 1)$ which have a continued fraction expansion begins by $a_1, a_2, \cdots, a_n$.
- Gauss Measure : The Gauss measure G on $[0, 1)$ is given by

$$dG(x) = \frac{1}{\log 2}\frac{1}{x+1}dx.$$

T preserves Gauss measure G and is ergodic with respect to G.

Relation to Diophantine approximation

Dirichlent's Theorem, 1842 : Suppose $x \in [0,1)\backslash\mathbb{Q}$. Then there exists infinitely many pairs p, q of relatively prime integers such that

$$|x - \frac{p}{q}| < \frac{1}{q^2}.$$

Khintchine's Theorem, 1924 : Let $\varphi : \mathbb{R} \to \mathbb{R}_+$ be a continuous function such that $x^2\varphi(x)$ is not increasing. Then the set

$$\{x \in [0,1) : \ |x - \frac{p}{q}| < \varphi(q) \text{ i.o. } \frac{p}{q}\}$$

has Lebesgue measure zero if $\sum\limits_{q=1}^{\infty} q\varphi(q)$ converges and has full Lebesgue measure otherwise.

Jarnik, 1929, 1931, Besicovitch, 1934 : For any $\beta > 2$,

$$\dim_{\mathrm{H}} \left\{ x \in [0,1) : \ |x - \frac{p}{q}| < \frac{1}{q^\beta} \text{ i.o. } \frac{p}{q} \right\} = \frac{2}{\beta}.$$

Bugeaud, 2003 Bugeaud studied the sets of exact approximation order by rational numbers which significantly strengthens the result of Jarnik and Besicovitch.

Lagrange's Theorem :

$$|x - \frac{p}{q}| < \frac{1}{2q^2} \Longrightarrow \frac{p}{q} = \frac{p_n(x)}{q_n(x)} \text{ for some } n.$$

Moreover,

$$\frac{1}{(a_{n+1}(x)+2)q_n(x)^2} \le |x - \frac{p_n(x)}{q_n(x)}| \le \frac{1}{a_{n+1}(x)q_n(x)^2}.$$

Thus for any $\beta > 2$,

$$
\begin{aligned}
& \{x \in [0,1): \ |x - \frac{p}{q}| < \frac{1}{q^\beta} \text{ i.o. } \frac{p}{q}\} \\
= \ & \{x \in [0,1): \ |x - \frac{p_n(x)}{q_n(x)}| < \frac{1}{q_n(x)^\beta} \text{ i.o. } n\} \\
\sim \ & \{x \in [0,1): \ a_{n+1}(x) > q_n(x)^{\beta-2} \text{ i.o. } n\}
\end{aligned}
$$

We concluding : The growth speed of the partial quotients $\{a_n(x)\}_{n\geq 1}$ reveals the speed how well a point can be approximated by rationals. In this talk, we shall concentrate on the sets of points whose partial quotients $\{a_n(x)\}_{n\geq 1}$ have different growth speed.

The growth speed of $\{a_n(x)\}_{n\geq 1}$

Borel-Bernstein Theorem, 1912 : Let ϕ be an arbitrary positive function defined on natural numbers $\mathbb{N}$ and

$$F(\phi) = \{x \in [0,1) : \ a_n(x) \geq \phi(n) \ \ i.o.\}.$$

If the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ converges, then $\mathcal{L}^1(F(\phi)) = 0$. If the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ diverges, then $\mathcal{L}^1(F(\phi)) = 1$.

- Good, 1941 :

$$\dim_{\mathrm{H}}\{x : a_n(x) \to \infty, \text{ as } n \to \infty\} = \frac{1}{2}.$$

- Hirst, 1970 : For any $a > 1$,

$$\dim_{\mathrm{H}}\{x \in [0,1) : \ a_n(x) \geq a^n \text{ for any } n \geq 1\} = \frac{1}{2}.$$

- Luczak, 1997 : For any $a > 1$, $b > 1$,

$$\begin{aligned} & \dim_{\mathrm{H}}\{x : a_n(x) \geq a^{b^n}, \forall\, n \in \mathbb{N}\} \\ = \ & \dim_{\mathrm{H}}\{x : a_n(x) \geq a^{b^n}, \text{ i.o. } n \in \mathbb{N}\} = \frac{1}{b+1}. \end{aligned}$$

- Multifractal analysis :

M. Pollicott, B. Weiss ; M. Kessebohmer, B. Stratmann ; A. Fan, L. Liao, Wang, Wu ; T. Jordan ; G. Iommi ; $\cdots$

Our intention

For general ψ, what are dimensions of the sets

$$E_{all}(\psi) = \Big\{x : a_n(x) \geq \psi(n), \forall\ n \in \mathbb{N}\Big\}$$

and

$$F_{i.o.}(\psi) = \Big\{x : a_n(x) \geq \psi(n),\ \text{i.o. } n \in \mathbb{N}\Big\}?$$

Results :

Wang and Wu (2008)

$$\dim_{\mathsf{H}} E_{all}(\psi) = \frac{1}{1+b}, \qquad b = \limsup_{n\to\infty} \frac{\log\log\psi(n)}{n},$$

Wang and Wu (2008)

Let

$$\liminf_{n\to\infty} \frac{\log\psi(n)}{n} = \log B, \quad \liminf_{n\to\infty} \frac{\log\log\psi(n)}{n} = \log b.$$

- when $1 \leq B < \infty$,

$$\dim_{\mathsf{H}} F_{i.o.}(\psi) = \inf\{s \geq 0 : P(-s(\log|T'| + \log B)) \leq 0\},$$

- when $B = \infty$, $\dim_{\mathsf{H}} F_{i.o.}(\psi) = 1/(1+b)$, Good (1941) also considered the general set. For any $B > 1$, let

$$d_B = \dim_{\mathrm{H}}\{x \in [0,1) : \ a_n(x) \geq B^n \ \ i.o.\}.$$

He gave the bound estimation on its Hausdorff dimension only but not the exact value. For any $s > \frac{1}{2}$, let $\theta(s) = 2^{-s}3^{-s} + 3^{-s}4^{-s} + \cdots$. Let s_0 satisfy $\theta(s_0) = 1$.

For any $s > 1$, let $\xi(s) = \sum_{n=1}^{\infty} n^{-s}$, Riemann's zeta function.

- Good, 1941 :
  - If $\frac{1}{2} < s < s_0$ and $B^{4s} < \theta(s)$, then $d_B \geq s$.
  - If $s > \frac{1}{2}$ and $B^s \geq \xi(2s)$, then $d_B \leq s$.
  - $d_B \to \frac{1}{2}$ as $B \to \infty$.

Sketch of Proof : $\{x : a_n(x) \geq B^n,\ i.o.n\}$

Upper bound :

For any $(a_1, \cdots, a_n)$, define *basic interval* :

$$J_n(a_1, \cdots, a_n) := \Big\{x : a_1(x) = a_1, \cdots, a_n(x) = a_n, a_{n+1}(x) \geq B^{n+1}\Big\}.$$

Then we get a cover of $F_{i.o.} = \{x : a_n(x) \geq B^n,\ i.o.n\}$:

$$F_{i.o.} \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{a_1, \cdots, a_n \in \mathbb{N}} J_n(a_1, \cdots, a_n).$$

So, the Hausdorff measure can be estimated as

$$\mathcal{H}^s(F_{i.o.}) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{a_1, \cdots, a_n \in \mathbb{N}} \left(\frac{1}{q_n^2(a_1, \cdots, a_n) B^{n+1}} \right)^s.$$

Lower bound :

Define a Cantor subset :

- $\{n_k\}$ a largely sparse subsequence of $\mathbb{N}$,
- α a large integer.
- define Cantor subset

$$E_B(\alpha) = \left\{ x : \begin{cases} 1 \le a_n(x) \le \alpha, & \text{when } n \ne n_k\,; \\ a_n(x) \in [B^n, 2B^n), & \text{when } n = n_k. \end{cases} \right\}$$

Result :

$$\dim_{\mathsf{H}} E_B(\alpha) = \inf \left\{ s \ge 0 : P_\alpha(-s(\log |T'| + \log B)) = 0 \right\},$$

where

$$P_\alpha(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{1 \le a_1, \cdots, a_n \le \alpha} \exp(\psi(x) + \cdots + \psi(T^{n-1}x)).$$

Lemma (Mauldin & Urbanski, 1996)

Let $f : [0,1] \to \mathbb{R}$ *be a function satisfying the tempered distortion condition, then*

$$P(f) = \sup\{P_\Sigma(f) : \Sigma \textit{ is } T \textit{ invariant subsystem}\}.$$

$\varphi : [0,1] \to \mathbb{R}$, write $S_n(\varphi)(x) = \psi(x) + \cdots + \psi(T^{n-1}x)$,

$$var_n(\varphi) := \sup_{a_1, a_2, \cdots, a_n} \{|\varphi(x) - \varphi(y)| : \ x, y \in I(a_1, a_2, \cdots, a_n)\}$$

- Tempered Distortion Property :

$$var_1(\varphi) < \infty \text{ and } \lim_{n \to \infty} \frac{1}{n} var_n S_n(\varphi) = 0.$$

Generalizations

Note that

$$\frac{1}{a_{n+1}(x)+1} \le T^n x \le \frac{1}{a_{n+1}(x)}.$$

So the growth rate of $a_n(x)$ can also be given by the speed that $T^n x$ approximate the point 0.

Shrinking target : For fixed y, how about the size of the set

$$\Big\{x : |T^n x - y| \le \psi(n,x), \text{ i.o. } n \in \mathbb{N}\Big\}?$$

In particular, let $f : [0,1] \to \mathbb{R}_+$, define

$$S_y(f) = \big\{x \in [0,1] : |T^n x - y| \le e^{-S_n f(x)}, \text{ i.o. } n\big\},$$

how about the size of $S_y(f)$?

From the viewpoint of dynamical system

Suppose we have a dynamical system (X, T, μ), where μ is a T-invariant ergodic probability measure. Let $A \subset X$ such that $\mu(A) > 0$. Ergodic property implies that

$$\mu(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} T^{-n}(A)) = 1,$$

that is, μ almost every $x \in X$ will visit A an infinite number of times. This raises the question of what happens when we allow A to shrink with respect to time. How does the size of $\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} T^{-n}(A(n))$ depend upon the sequence $\{A(n)\}_{n \geq 1}$?

The shrinking target problem initialed by Hill and Velani (1995) which concerns "what happens if the target shrinks with the time and more generally if the target also moves around with the time."

From the viewpoint of Diophantine approximation

Sets of the form $S_y(f)$ arise naturally in Diophantine approximation. Given $\beta > 2$, let

$$
\begin{aligned}
J(\beta) &= \{x \in [0,1) : \ |x - \frac{p}{q}| < \frac{1}{q^\beta} \text{ i.o. } \frac{p}{q}\} \\
&= \{x \in [0,1) : \ |x - \frac{p_n(x)}{q_n(x)}| < \frac{1}{q_n(x)^\beta} \text{ i.o. } n\} \\
&\sim \{x \in [0,1) : \ a_{n+1}(x) > q_n(x)^{\beta-2} \text{ i.o. } n\}.
\end{aligned}
$$

Choose $f = \log |T'|$. In this case, we have $S_n(f) = \log |(T^n)'|$ and $|(T^n)'(x)| \approx q_n(x)^2$.
For each $\alpha > 2$, let $f_\alpha = (\frac{\alpha}{2} - 1)f$. Then for any $2 < \alpha < \beta < \gamma$, it is easy to see

$$S_0(f_\gamma) \subset J(\beta) \subset S_0(f_\alpha).$$

Thus

$$\dim_{\mathrm{H}} J(\beta) = \frac{2}{\beta} \Longleftrightarrow \dim_{\mathrm{H}} S_0(f_\beta) = \frac{2}{\beta}, \ \text{for any } \beta > 2.$$

Our result

The shrinking target problem in the system of continued fractions :

Li, Wang, Wu and Xu(2010)

Let $f : [0,1] \to \mathbb{R}_+$ be a function satisfying the tempered distortion condition, then

$$\dim_{\mathrm{H}} S_y(\psi) = \inf\left\{s \geq 0 : P(-s(\log|T'| + f)) \leq 0\right\},$$

For $B \geq 1$, take $f = \log B$, we have

Wang, Wu(2008)

$$\begin{aligned} & \dim_{\mathrm{H}}\{x \in [0,1) : \ a_n(x) \geq B^n \ \ i.o.\} \\ = \ & \inf\left\{s \geq 0 : P(-s(\log|T'| + \log B)) \leq 0\right\}. \end{aligned}$$

Lemma (A transfer principle : Relation of cylinders and balls)

Let $B(z,r)$ be a ball with center $z \in [0,1]$ and radius $0 < r < e^{-4}$. Then there exist integers $t \le -4\log r$, $b_1, \cdots, b_{t-1}$ and $\underline{b}_t, \overline{b}_t$ such that $3 \le \underline{b}_t < \overline{b}_t$ and the family

$$\mathbb{G} = \left\{ I(b_1, \cdots, b_{t-1}, b_t) : \underline{b}_t < b_t \le \overline{b}_t \right\}$$

satisfies the following three conditions.

(1) *All the cylinders in $\mathbb{G}$ are of comparable length :*

$$1/24 \le \frac{|I(b_1, \cdots, b_{t-1}, b_t)|}{|I(b_1, \cdots, b_{t-1}, b'_t)|} \le 24, \quad \text{for all } \underline{b}_t < b_t, b'_t \le \overline{b}_t.$$

(2) *All the cylinders I in $\mathbb{G}$ are contained in the ball $B(z,r)$.*

(3) *The cylinders in $\mathbb{G}$ pack the ball $B(z,r)$ sufficiently ; that is*

$$2r \ge \sum_{\underline{b}_t < b_t \le \overline{b}_t} \left| I(b_1, \cdots, b_{t-1}, b_t) \right| \ge \frac{r}{4^6}.$$

Localized Jarnik-Bescovitch theorem

Suppose $\tau : [0,1] \to [2,\infty)$ be a continuous function, let

$$J(\tau) = \Big\{ x \in [0,1) : \ |x - \frac{p}{q}| < \frac{1}{q^{\tau(x)}} \text{ i.o. } \frac{p}{q} \Big\}.$$

- Barral and Seuret, 2011 :

$$\dim_{\mathrm{H}} J(\tau) = \frac{2}{\min\{\tau(x) : \ x \in [0,1]\}},$$

this result extends the classical Jarnik-Bescovitch theorem.

Let $\eta : [0,1] \to [0,\infty)$ be a continuous function, define

$$L_\eta(f) := \Big\{ x : a_{n+1}(x) \geq e^{\eta(x) S_n f(x)}, \text{ i.o. } n \in \mathbb{N} \Big\}$$

Remark : When $\eta(x) = \frac{\tau(x)}{2} - 1$ and $f(x) = \log |T'(x)|$, then

$$L_\eta(f) \sim J(\tau).$$

Wang, Wu, Xu(2012)

Let $\eta : [0,1] \to [0,\infty)$ be a continuous function and $f : [0,1] \to \mathbb{R}_+$ be a function satisfying the tempered distortion condition, then

$$\dim_{\mathrm{H}} L_\eta(f) = \inf \Big\{ s \geq 0 : P\Big(-s\big(\log|T'| + f \min_{x \in [0,1]} \eta(x)\big)\Big) \leq 0 \Big\}.$$

General settings of the shrinking target problems : How about the size of the following type of sets ?

- Fixed target problems :

$$S_y(\psi) := \Big\{x : |T^n x - y| \le \psi(n, x),\ \text{i.o. } n \in \mathbb{N}\Big\}$$

- Recurrence properties :

$$R(\psi) := \Big\{x : |T^n x - x| \le \psi(n, x),\ \text{i.o. } n \in \mathbb{N}\Big\}$$

- Covering problems :

$$C(\psi) := \Big\{y : |T^n x_0 - y| \le \psi(n, x_0),\ \text{i.o. } n \in \mathbb{N}\Big\}$$

Known Results : Size in measure

Boshernitzan ; Barreira, Saussol ; Chernov and Kleinbock ; Chazottes ; Saussol ; Fayad ; Galatalo ; J. Tseng ; Kim ; Fernàndez ; Meliàn, Pestana ; . . .

Known Results : Size in dimension

Fixed target problems :

- Hill, Velani 1995, 1997. T an expanding rational map on Riemann sphere and J its Julia sets.
- Hill Velani 1999. (X, T), X n-dimensional torus and T a linear operator given by a matrix with integer coefficients.
- Urbański, 2002 countable expanding Markov map : partial result.
- Stratmann and Urbański, 2002 Parabolic rational maps on Julia set.
- Fernàndez, Meliàn, Pestana 2007. General expanding Markov systems.
- H. Reeve, 2011 countable expanding Markov map.

Recurrence properties :

- Tan, Wang, 2011. $([0,1], T_\beta)$ the system of β-expansion.

$$\dim_{\mathsf{H}} \left\{x \in J : |T^n x - x| \le e^{-S_n f(x)},\ \text{i. o.}\right\} = \inf\{s : P(-sf) \le 0\}.$$

- Seuret, Wang, 2012. Infinite conformal IFS.

Covering problems :

- Schmeling & Troubetzkoy 2003, Bugeaud 2003, Fan & Wu 2006
Irrational rotation :

$$\{y \in [0,1] : |n\alpha - y| < \varphi(n) \ i.o.\}$$

- Fan, Schmeling, Troubetzkoy 2007 Doubling map.

$$\left\{y \in [0,1] : |y - 2^n x| < 1/n^{\beta} \ i.o.\right\}.$$

- Liao, Seuret 2010 : General expanding Markov maps.

Thanks for your attention !