Packing Dimension Results for Anisotropic Gaussian Random Fields

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(Based on a joint work with Anne Estrade and Yimin Xiao)

Outline

Fractal Dimensions

- In charactering roughness or irregularity of stochastic processes and random fields [cf. Taylor (1986) and Xiao (2004) for Markov processes, and Adler (1981), Kahane (1985), Khoshnevisan (2002) and Xiao (2007, 2009a) for Gaussian processes and fields]
- In statistical analysis of the processes and fields [cf. Gneiting, Sevcikova and Percival (2012) and references therein]

Image and Graph of an (N, d) Random Field

Let $\{X(t), \ t \in \mathbb{R}^N\}$ be an (N, d) random field, and $E \subseteq \mathbb{R}^N$ be a Borel set. Define

- $X(E) = \{X(t), \ t \in E\}$
- $\mathrm{Gr}X(E) = \{(t, X(t)), \ t \in E\}$

Dimension Results: Fractional Brownian Motion

If X is a fractional Brownian motion,

- $\dim_{\mathrm{H}} X\left([0,1]^N\right) = \dim_{\mathrm{P}} X\left([0,1]^N\right)$
- For an arbitrary E, the Hausdorff dimension and the packing dimension results of $X(E)$ (when $\alpha d < N$) can be different [cf. Talagrand and Xiao (1996)]

Packing Dimension Profile

- First, by Falconer and Howroyd (1997), for computing the packing dimension of orthogonal projections, based on potential theoretical approach.
- Later, Howroyd (2001) defined another packing dimension profile from box-counting dimension point of view.
- Khoshnevisan and Xiao (2008), via the establishing of a new property of fractional Brownian motion and a probabilistic argument, proved that these two definitions of packing dimension profile are the same.
- Recently, Khoshnevisan, Schilling and Xiao (2012) extended the notion of packing dimension profiles in order to determine the packing dimension of an arbitrary image of a general Lévy process. Zhang (2012) further extended their notion to higher dimensional case for the image of an additive Lévy process.

Packing Dimension of $X(E)$

$\dim_{\mathrm{P}} X(E)$ is determined by the packing dimension profiles introduced by Falconer and Howroyd (1997) [cf. Xiao (1997)]

$$\dim_{\mathrm{P}} X(E) = \frac{1}{\alpha} \mathrm{Dim}_{\alpha d} E,$$

where α is the Hurst index of the fractional Brownian motion, and $\mathrm{Dim}_s E$ is the packing dimension profile of E.

Dimension Results: Approximately Isotropic Gaussian Fields [Xiao (2007, 2009b)]

- $X(t) = (X_1(t), \ldots, X_d(t))\,, \ \forall t \in \mathbb{R}^N$
- $\mathbb{E}\left[(X_0(s) - X_0(t))^2\right] \asymp \phi^2(\|t-s\|), \qquad \forall\, s, t \in [0,1]^N$ (Approximately isotropic)
- Upper index of ϕ at 0 is defined by

$$\alpha^* = \inf\left\{\beta \geq 0 : \lim_{r\to 0} \frac{\phi(r)}{r^\beta} = \infty\right\} \tag{1}$$

- Lower index of ϕ at 0 is defined by

$$\alpha_* = \sup\left\{\beta \geq 0 : \lim_{r\to 0} \frac{\phi(r)}{r^\beta} = 0\right\} \tag{2}$$

- **Remark:** There are many interesting examples of Gaussian random fields with stationary increments with $\alpha_* < \alpha^*$. [cf. Xiao (2007), Estrade, Wu and Xiao (2011)]

Dimension Results: Approximately Isotropic Gaussian Fields [Xiao (2007, 2009b)]

- Hausdorff dimension results [cf. Xiao (2007)]

$$\dim_{\rm H} X\big([0,1]^N\big) = \min\left\{d,\ \frac{N}{\alpha^*}\right\}, \qquad \text{a.s.} \tag{3}$$

$$\dim_{\rm H} {\rm Gr} X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha^*},\ N + (1-\alpha^*)d\right\}, \qquad \text{a.s.} \tag{4}$$

- Packing dimension results [cf. Xiao 2009b]

$$\dim_{\rm P} X\big([0,1]^N\big) = \min\left\{d,\ \frac{N}{\alpha_*}\right\}, \qquad \text{a.s.} \tag{5}$$

$$\dim_{\rm P} {\rm Gr} X\big([0,1]^N\big) = \min\left\{\frac{N}{\alpha_*},\ N+(1-\alpha_*)d\right\}, \qquad \text{a.s.} \tag{6}$$

Dimension Results: (Approximately) Isotropic Random Fields [Shieh and Xiao (2010)]

Recently, under some mild conditions, Shieh and Xiao (2010) determine the Hausdorff and packing dimensions of the image measure $\mu_X$and image set $X(E)$. Their results can be applied to Gaussian random fields, self-similar stable random fields with stationary increments, real harmonizable fractional Lévy fields and the Rosenblatt process.

This Talk

We derive packing dimension results for a class of anisotropic Gaussian random fields satisfying:
Condition C: For every compact interval $T \subset \mathbb{R}^N$, there exist positive constants δ_0 and $K \geq 1$ such that

$$K^{-1}\,\phi^2(\rho(s,t)) \leq \mathbb{E}\Big[\big(X_0(t) - X_0(s)\big)^2\Big] \leq K\,\phi^2(\rho(s,t)) \qquad (7)$$

for all $s, t \in T$ with $\rho(s,t) \leq \delta_0$, where ρ is an anisotropic metric (on $\mathbb{R}^N$) defined by, for some $H_j \in (0,\ 1)$, $j = 1, \ldots, N$

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall s, t \in \mathbb{R}^N \qquad (8)$$

Modulus of Continuity [cf. Dudley (1973)]

If X_0 satisfies Condition C, then for every compact interval $T \subset \mathbb{R}^N$, there exists a finite constant K such that

$$\limsup_{\delta \to 0} \frac{\sup_{s,t \in T: \rho(s,t) \le \delta} |X_0(s) - X_0(t)|}{f(\delta)} \le K, \qquad \text{a.s.}, \tag{9}$$

where $f(h) = \phi(h) \big| \log \phi(h) \big|^{1/2}$.

Packing Dimension and Packing Dimension Profile on $(\mathbb{R}^N, \rho)$

For studying Hausdorff and packing dimension results of the images of anisotropic Gassian fields, the notions of Hausdorff dimension [cf. Wu and Xiao (2007, 2009)] and packing dimension [cf. Estrade, Wu and Xiao (2011)] on $(\mathbb{R}^N, \rho)$ are needed.

In the following, we extend the notions of packing dimension of a set [cf. Tricot (1982)], packing dimension of a measure [cf. Tricot and Taylor (1985)] and packing dimension profile [cf. Falconer and Howroyd (1997)] to metric space $(\mathbb{R}^N, \rho)$.

Remark: When $H_1 = \cdots = H_N$, they are equivalent to the notions in Euclidean space $\mathbb{R}^N$.

Packing Measure in Metric ρ

- $B_\rho(x,r) := \{y \in \mathbb{R}^N : \rho(y,x) < r\}$.
- β-dimensional packing measure of E in the metric ρ is defined by

$$\mathcal{P}_\rho^\beta(E) = \inf\left\{\sum_n \overline{\mathcal{P}_\rho^\beta}(E_n) : \quad E \subseteq \bigcup_n E_n\right\}, \tag{10}$$

where

$$\overline{\mathcal{P}_\rho^\beta}(E) = \lim_{\delta\to 0} \sup\left\{\sum_{n=1}^{\infty}(2r_n)^\beta : \{B_\rho(x_n, r_n)\} \text{ are disjoint, }\right\}. \tag{11}$$

Packing Dimension in Metric ρ

- $$\dim_{\mathrm{P}}^{\rho} E = \inf\left\{\beta > 0 : \ \mathcal{P}_{\rho}^{\beta}(E) = 0\right\}. \qquad (12)$$
- We have, as an extension of a result of Tricot (1982),

$$\dim_{\mathrm{P}}^{\rho} E = \inf\left\{\sup_{n} \overline{\dim}_{\mathrm{B}}^{\rho} E_n : \ E \subseteq \bigcup_{n=1}^{\infty} E_n\right\}, \qquad (13)$$

where

$$\overline{\dim}_{\mathrm{B}}^{\rho} E = \limsup_{\varepsilon \to 0} \frac{\log N_{\rho}(E, \varepsilon)}{-\log \varepsilon}.$$

Some Properties of the Packing Dimension in Metric ρ

- It is σ-stable.
- Denote $Q := \sum_{j=1}^{N} H_j^{-1}$, we have

$$0 \le \dim_{\mathrm{H}}^{\rho} E \le \dim_{\mathrm{P}}^{\rho} E \le \overline{\dim}_{\mathrm{B}}^{\rho} E \le Q, \tag{14}$$

and $\dim_{\mathrm{H}}^{\rho} E = \dim_{\mathrm{P}}^{\rho} E$, if E has nonempty interior.

Packing Dimension of a Measure in Metric ρ

- $$\dim_{\rm P}^{\rho} \mu = \inf\{\dim_{\rm P}^{\rho} E : \ \mu(E) > 0 \text{ and } E \subseteq \mathbb{R}^N \text{ is a Borel set}\}. \tag{15}$$

- A characterization of $\dim_{\rm P}^{\rho} \mu$ in terms of the local dimension of μ, obtained by applying Lemma 4.1 of Hu and Taylor (1994) to $\dim_{\rm P}^{\rho}$:

$$\dim_{\rm P}^{\rho} \mu = \sup\left\{\beta > 0 : \liminf_{r\to 0} \frac{\mu\big(B_\rho(x, r)\big)}{r^\beta} = 0 \ \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}. \tag{16}$$

Packing Dimension Profile of a Measure in Metric ρ

- s-dimensional packing dimension profile of μ in metric ρ as

$$\mathrm{Dim}_s^{\rho}\mu = \sup\left\{\beta \geq 0 : \ \liminf_{r\to 0} \frac{F_{s,\rho}^{\mu}(x,r)}{r^{\beta}} = 0 \ \text{ for } \mu\text{-a.a. } x \in \mathbb{R}^N\right\}, \tag{17}$$

where, for any $s > 0$, $F_{s,\rho}^{\mu}(x,r)$ is the s-dimensional potential of μ in metric ρ defined by

$$F_{s,\rho}^{\mu}(x,r) = \int_{\mathbb{R}^N} \min\left\{1, \ \frac{r^s}{\rho(x,y)^s}\right\} d\mu(y). \tag{18}$$

A Property

$$0 \le \mathrm{Dim}_s^{\rho}\mu \le s \text{ and } \mathrm{Dim}_s^{\rho}\mu = \dim_{\mathrm{P}}^{\rho}\mu \text{ if } s \ge Q. \qquad (19)$$

Furthermore, $\mathrm{Dim}_s^{\rho}\mu$ is continuous in s.

Packing Dimension Profile of a Set in Metric ρ

- s-dimensional packing dimension profile of E in the metric ρ is defined by

$$\mathrm{Dim}_s^{\rho} E = \sup\left\{\mathrm{Dim}_s^{\rho}\mu : \ \mu \in \mathcal{M}_c^+(E)\right\}. \tag{20}$$

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$$0 \le \mathrm{Dim}_s^{\rho} E \le s \quad \text{and} \quad \mathrm{Dim}_s^{\rho} E = \dim_{\mathrm{P}}^{\rho} E \quad \text{if} \ \ s \ge Q. \tag{21}$$

Outline

Packing Dimension of $X\left([0,\ 1]^N\right)$ [Estrade, Wu and Xiao (2011)]

Let X be an anisotropic Gaussian field satisfying Condition C, with ϕ is such that $0 < \alpha_* \le \alpha^* < 1$ and satisfies one of the following conditions:

$$\int_0^N \left(\frac{1}{\phi(x)}\right)^{d-\varepsilon} x^{Q-1}\, dx \le K \qquad (22)$$

or

$$\int_1^{N/a} \left(\frac{\phi(a)}{\phi(ax)}\right)^{d-\varepsilon} x^{Q-1}\, dx \le K\, a^{-\varepsilon} \quad \text{for all } \ a \in (0,1]. \qquad (23)$$

Then with probability 1,

$$\dim_{\rm P} X\left([0,1]^N\right) = \min\left\{d; \quad \frac{Q}{\alpha_*}\right\}. \qquad (24)$$

Packing Dimension of $X\left([0,\ 1]^N\right)$ (Proof)

- Upper bound: The modulus of continuity of X and a covering argument.
- Lower bound: Potential theoretic approach to packing dimension of finite Borel measures.

Outline

1. Introduction
2. Packing Dimension and Packing Dimension Profile on $(\mathbb{R}^N, \rho)$
3. Packing Dimension Results for Anisotropic Gaussian Fields
   - Packing Dimension of $X\left((0,1)^N\right)$
   - Packing Dimension of $X(E)$

Packing Dimension of μ_X [Estrade, Wu and Xiao (2011)]

For any finite Borel measure μ on $\mathbb{R}^N$, with probability 1,

$$\frac{1}{\alpha^*}\mathrm{Dim}^{\rho}_{\alpha^* d}\mu \leq \dim_{\mathrm{P}}\mu_X \leq \frac{1}{\alpha_*}\mathrm{Dim}^{\rho}_{\alpha_* d}\mu. \tag{25}$$

Packing Dimension of $X(E)$ (Proof)

- First inequality: Potential theoretic approach to packing dimension of finite Borel measures.
- Second inequality: The modulus continuity of X.

Packing Dimension of $X(E)$ [Estrade, Wu and Xiao (2011)]

- If $0 < \alpha_* = \alpha^* < 1$, then for every analytic set $E \subseteq [0,1]^N$, we have that

$$\dim_{\rm P} X(E) = \frac{1}{\alpha} {\rm Dim}^{\rho}_{\alpha d} E \qquad \text{a.s.},$$

where $\alpha := \alpha^* = \alpha_*$.

- **Remark:** The problems for finding $\dim_{\rm H} X(E)$ and $\dim_{\rm P} X(E)$ are still open when $\alpha_* \neq \alpha^*$

Thank You!