Open set condition for self-similar structure

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Outline

Shift space.

Let $S = \{1, 2, \ldots, N\}$ be a finite set with N elements. A word over S is a sequence $\mathbf{w} = w_1 w_2 \ldots w_n \ldots$ with $w_n \in S$ for each n. We denote by

$$S^n = \{w_1 w_2 \ldots w_n : w_n \in S, 1 \leq j \leq n\}$$

the set of words of length n and denote by $|\mathbf{w}| = n$ the length of $\mathbf{w} \in S^n$. Let $S^* = \bigcup_{n \geq 0} S^n$ be the set of finite words, where the empty word ε is of length 0. The set of infinite words $S^{\mathbb{N}}$ is called the **shift space** with N-symbols. For each $a \in S$, define a map $\sigma_a : S^{\mathbb{N}} \to S^{\mathbb{N}}$ by

$$\sigma_a(w_1 w_2 \ldots w_n \ldots) = a w_1 w_2 \ldots w_n \ldots.$$

We also define shift map $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ by

$$\sigma(w_1 w_2 \ldots w_n \ldots) = w_2 \ldots w_n \ldots.$$

Self-similar set v.s. self-similar structure.

Definition (Self-similar set)

For each $a \in S = \{1,2,\ldots,N\}$, map $\phi_a : \mathbb{R}^n \to \mathbb{R}^n$ is a similitude. The self-similar set is the unique compact set $\mathscr{K} \subset \mathbb{R}^n$ satisfying $\mathscr{K} = \phi_1(\mathscr{K}) \cup \cdots \cup \phi_N(\mathscr{K})$.

Fact

(1) Compact set $\mathscr{K} \subset \mathbb{R}^n$;
(2) ϕ_a is a similitude.

Definition (Self-similar structure)

Let $\mathscr{K}$ be a compact metric space. For each $a \in S = \{1,2,\ldots,N\}$, map $\psi_a : \mathscr{K} \to \mathscr{K}$ is a continuous injection. Then, $(\mathscr{K}, S, \{\psi_a\}_{a\in S})$ is called a self-similar structure if there exists a continuous surjection $\pi : S^{\mathbb{N}} \to \mathscr{K}$ such that $\psi_a \circ \pi = \pi \circ \sigma_a$ for every $a \in S$, where $\sigma_a(w_1 w_2 \ldots) = a w_1 w_2 \ldots$.

Fact

(1) $\mathscr{K}$ itself is a compact metric space;
(2) ψ_a is only a continuous injection.

Self-similar structure describes the topology.

If $\mathscr{K}$ is a self-similar set with similitudes $\phi_1,\ldots,\phi_N$, then $(\mathscr{K},\{1,\ldots,N\},\{\phi_a\}_{a=1}^N)$ is a self-similar structure.

Example (Interval)

Let $S=\{1,2\}$, $\phi_1(x)=\frac{1}{2}x$ and $\phi_2(x)=\frac{1}{2}x+\frac{1}{2}$. Then self-similar set $\mathscr{I}=[0,1]$. Denote by $\mathscr{I}_{\mathbf{v}}=\phi_{\mathbf{v}}(\mathscr{I})=\phi_{v_1}\circ\phi_{v_2}\circ\cdots\circ\phi_{v_n}(\mathscr{I})$ for word $\mathbf{v}=v_1v_2\ldots v_n\in S^n$. Then

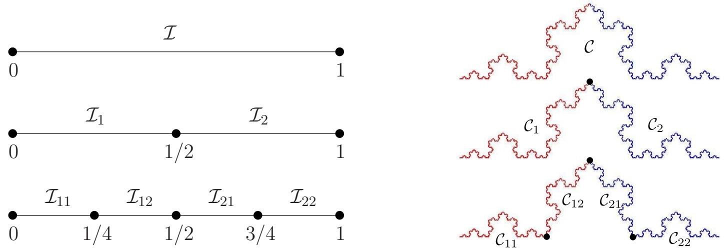
$$\mathscr{I}=\mathscr{I}_1\cup\mathscr{I}_2=\mathscr{I}_{11}\cup\mathscr{I}_{12}\cup\mathscr{I}_{21}\cup\mathscr{I}_{22}=\bigcup_{\mathbf{v}\in S^n}\mathscr{I}_{\mathbf{v}}.$$

For any $\mathbf{w}=w_1w_2\ldots w_n\ldots\in S^{\mathbb{N}}$, the intersection $\bigcap_{n\geq 0}\mathscr{I}_{w_1w_2\ldots w_n}$ contains only one point. Thus the map $\pi_{\mathscr{I}}:S^{\mathbb{N}}\to\mathscr{I}$ is well defined by $\{\pi_{\mathscr{I}}(\mathbf{w})\}=\bigcap_{n\geq 0}\mathscr{I}_{w_1w_2\ldots w_n}$. Furthermore, $\phi_a\circ\pi=\pi\circ\sigma_a$ for each $a\in S$, that is, $(\mathscr{I},\{1,2\},\{\phi_1,\phi_2\})$ is a self-similar structure.

Interval and Koch curve have the same self-similar structure.

Example (Koch curve)

Let $S=\{1,2\}$, $\phi_1(x)=(-\frac{1}{2}-\frac{i}{2\sqrt{3}})x+(\frac{1}{2}+\frac{i}{2\sqrt{3}})$ and $\phi_2(x)=(-\frac{1}{2}+\frac{i}{2\sqrt{3}})x+1$. Then self-similar set $\mathscr{C}$ is the Koch curve. In the same way with interval $\mathscr{I}=[0,1]$, we can defined a surjection $\pi_{\mathscr{C}}:S^{\mathbb{N}}\to\mathscr{C}$ such that $(\mathscr{C},\{1,2\},\{\phi_1,\phi_2\})$ is a self-similar structure. Note that $\pi_{\mathscr{C}}\circ\pi_{\mathscr{I}}^{-1}$ is a homeomorphism between $\mathscr{I}=[0,1]$ and Koch curve $\mathscr{C}$.



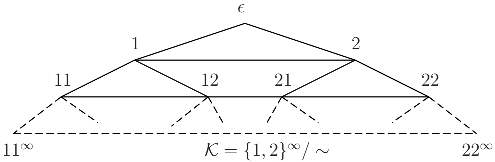
Compact set as quotient of shift space.

Example (Quotient space)

For two infinite words $\mathbf{w}, \mathbf{w}' \in S^{\mathbb{N}}$, define $\mathbf{w} \sim \mathbf{w}'$ if they are of forms

$$\mathbf{w} = \mathbf{u}12^{\infty} \text{ and } \mathbf{w}' = \mathbf{u}21^{\infty}$$

for some finite word $\mathbf{u} \in S^*$. Let $\mathscr{K} = S^{\mathbb{N}}/\sim$ be the quotient space with quotient map $\pi : S^{\mathbb{N}} \to \mathscr{K}$. Then the triple $(\mathscr{K}, \{1,2\}, \{\psi_1, \psi_2\})$ is a self-similar structure, where $\psi_a = \pi \circ \sigma_a \circ \pi^{-1}$, $a = 1, 2$. Note that $\pi \circ \pi_{\mathscr{I}}^{-1}$ is a homeomorphism between $\mathscr{I} = [0,1]$ and quotient space $\mathscr{K}$.



Open set condition for self-similar set.

Definition

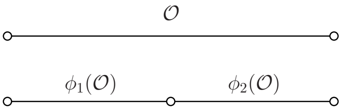
Let $\mathscr{K}$ be a self-similar set with similitudes $\phi_1,\ldots,\phi_N$. We say $\mathscr{K}$ fulfills open set condition if there is an open set $\mathscr{O}$ satisfying

(i) $\phi_a(\mathscr{O})\cap\phi_{a'}(\mathscr{O})=\emptyset$, for any $a\neq a'$;

(ii) $\phi_a(\mathscr{O})\subset\mathscr{O}$ for any $a\in S$.

Example

Consider the interval $\mathscr{I}=[0,1]$ with respect to iterated functions $\phi_1(x)=\frac{1}{2}x$ and $\phi_2(x)=\frac{1}{2}x+\frac{1}{2}$. Let $\mathscr{O}=(0,1)$. Then (i) $\phi_a(\mathscr{O})\cap\phi_{a'}(\mathscr{O})=\emptyset$ for any $a\neq a'$, and (ii) $\phi_a(\mathscr{O})\subset\mathscr{O}$ for any $a\in S$.



Equivalent conditions to open set condition

The open set condition is proved to equivalent each of the followings.

- **Positivity of α-dimensional Hausdorff measure.** More precisely, denote by r_a the contraction factor of ϕ_a for each $a \in S$. Let μ be the α-dimensional Hausdorff measure, where α is the similarity dimension satisfying $\sum_{a\in S} r_a^\alpha = 1$. Then, $\mathscr{K}$ fulfills open set condition is equivalent to $\mu(\mathscr{K}) > 0$.
  - Schief (1994, 1996) showed the equivalence for self-similar sets in $\mathbb{R}^n$ and in complete metric space.
  - Peres, Rams, Simon and Solomyak (2001) showed the equivalence for self-conformal sets.
- **Isolation of identity map.** That is, the identity map **id** is not an accumulation point of the set $\{\phi_{\mathbf{w}}^{-1} \circ \phi_{\mathbf{v}} : \mathbf{w}, \mathbf{v} \in S^*\}$.
  - Bandt and Graf (1992) showed it.

Other separation conditions

Definition

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure with nature map $\pi : S^{\mathbb{N}} \to \mathscr{K}$. Let

$$\mathscr{R}_{\mathscr{L}} = \bigcup_{a\neq a'} (\psi_a(\mathscr{K}) \cap \psi_{a'}(\mathscr{K}))$$

be the overlapping set. Define the critical set $\mathscr{C}_{\mathscr{L}} = \pi^{-1}(\mathscr{R}_{\mathscr{L}})$ and the post critical set $\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}})$.

(i) $\mathscr{L}$ is called **finitely ramified** if the overlapping set $\mathscr{R}_{\mathscr{L}}$ is finite;

(ii) $\mathscr{L}$ is called **post-critically finite** if the post critical set $\mathscr{P}_{\mathscr{L}}$ is finite.

Example

Consider the interval $\mathscr{I} = [0,1]$ with iterated functions $\psi_1(x) = \frac{1}{2}x$ and $\psi_2(x) = \frac{1}{2}x + \frac{1}{2}$. Then the overlapping $\mathscr{R}_{\mathscr{L}} = [0,1/2] \cap [1/2,1] = \{1/2\}$. Thus, the critical set $\mathscr{C}_{\mathscr{L}} = \pi^{-1}(\mathscr{R}_{\mathscr{L}}) = \{12^\infty, 21^\infty\}$ and the post critical set $\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}}) = \{1^\infty, 2^\infty\}$.

Relations between different separation conditions

Theorem (2007, Bandt and Rao)

Let $\mathscr{K}$ be a connected self-similar set in the plane. Then the finitely ramified condition implies open set condition.

In short, finitely ramified $\overset{(\text{connected } \mathscr{K}\subset\mathbb{R}^2)}{\Longrightarrow}$ open set condition.

Theorem (2008, Deng and Lau)

Let $\mathscr{K}$ be a self-similar set with respect to iterated functions $\psi_a(x) = M_a(x+c_a)$, $a \in S$, where $M_a = r_a O_a$, $0 < r_a < 1$, O_a is orthonormal matrix and $c_a \in \mathbb{R}^n$ for each $a \in S$. Suppose $\{M_a\}_{a\in S}$ is commensurable, that is, there exists a matrix M such that $M_a = M^{n_a}$ for some positive integer n_a, $a \in S$. Then the post-critically finite condition implies open set condition.

In short, p.c.f. $\overset{(\text{commensurable})}{\Longrightarrow}$ open set condition.

Separation conditions for self-similar structure

Definition (OSC)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure. We say $\mathscr{L}$ fulfills open set condition if there exists open set $\mathscr{O} \subset \mathscr{K}$ such that

(i) $\psi_a(\mathscr{O}) \cap \psi_{a'}(\mathscr{O}) = \emptyset$, for any $a \neq a'$;

(ii) $\psi_a(\mathscr{O}) \subset \mathscr{O}$ for any $a \in S$.

Definition (Finite preimage)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure with nature map $\pi : S^{\mathbb{N}} \to \mathscr{K}$. We say $\mathscr{L}$ fulfills finite preimage property if each point in $\mathscr{K}$ has only finitely many preimages under the map π.

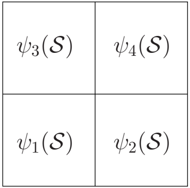
Basic observation

- p.c.f. $\Rightarrow$ finitely ramifed. Suppose a self-similar structure $\mathscr{L}$ is post-critically finite, that is, the post critical set $\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}})$ is finite. Therefore, critical set $\mathscr{C}_{\mathscr{L}}$ is finite, and thus the overlapping set $\mathscr{R}_{\mathscr{L}} = \pi(\mathscr{C}_{\mathscr{L}})$ is finite, which shows that $\mathscr{L}$ is finitely ramified.
- p.c.f. $\Rightarrow$ finite preimage. Suppose a self-similar structure $\mathscr{L}$ is post-critically finite, that is, the post critical set $\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}})$ is finite, and thus critical set $\mathscr{C}_{\mathscr{L}}$ is finite. Assume there is a point $x \in \mathscr{K}$ with infinite preimage set $\pi^{-1}(x)$. Let $\mathbf{w} \in S^n$ be the largest common prefix. Then critical set $\mathscr{C}_{\mathscr{L}} \supset \sigma^n(\pi^{-1}(x))$ is infinite, which is a contradiction.

Basic observation

Example (OSC $\not\Rightarrow$ finitely ramified)

Consider the square $\mathscr{S} = [0,1] \times [0,1]$ with iterated functions $\psi_1(x) = \frac{1}{2}x$, $\psi_2(x) = \frac{1}{2}x + \frac{1}{2}$, $\psi_3(x) = \frac{1}{2}x + \frac{i}{2}$ and $\psi_4(x) = \frac{1}{2}x + \frac{1}{2} + \frac{i}{2}$. Then self-similar structure $(\mathscr{S}, \{1,2,3,4\}, \{\psi_a\}_{a=1}^{4})$ fulfills open set condition but is not finitely ramified.



Characterizing OSC for self-similar structure.

In the topological viewpoint, compact set $\mathscr{K}$ is just the quotient space $S^{\mathbb{N}}/\sim$ with respect to equivalence relation $\sim$. The critical set $\mathscr{C}_{\mathscr{L}}$ turns out to be essential in charactering separation conditions for self-similar structure.

Theorem (Ni and Wen)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure. Then the followings are equivalent:

(a) *$\mathscr{L}$ fulfills open set condition;*

(b) *the post critical set $\mathscr{P}_{\mathscr{L}}$ is not dense in $S^{\mathbb{N}}$.*

Corollary

p.c.f. $\Rightarrow$ open set condition.

Characterizing finite preimage property.

Theorem (Ni and Wen)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure. Then the followings are equivalent:

(a) *The structure $\mathscr{L}$ fulfills finite preimage property;*

(b) $\limsup_{n\to\infty} \sigma^{-n}(\mathscr{C}_{\mathscr{L}}) = \emptyset$.

Fact

Recall that the equivalent condition to OSC is "$\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}})$ is not dense in cylinder $[\mathbf{v}]$ for each $\mathbf{v} \in S^*$". The difference between $\sigma^{-n}(\mathscr{C}_{\mathscr{L}})$ and $\sigma^n(\mathscr{C}_{\mathscr{L}})$ suggests that each of the two conditions "finite preimage" and "OSC" does not implies the other, which will be shown in the following.

Finitely ramified and finite preimage do not imply OSC.

Example (Finitely ramified + finite preimage $\not\Rightarrow$ OSC)

Define a equivalence relation $\sim$ on the shift space $\{1,2\}^{\mathbb{N}}$ as follows. Let $\mathbf{v}$ be an infinite word with all the finite word as factors. Two different words $\mathbf{w},\mathbf{w}' \in E^{\mathbb{N}}$ are equivalent if they are of forms

$$\mathbf{w} = \mathbf{u}2\mathbf{v} \text{ and } \mathbf{w}' = \mathbf{u}12^{\infty}, \quad \mathbf{u} \in \{1,2\}^*.$$

The complete metric space $\mathscr{K}$ is the quotient space $\{1,2\}^{\mathbb{N}}/\sim$ with quotient metric. Denoting by π the quotient map and $x = \pi(2\mathbf{v}) = \pi(12^{\infty})$, we obtain the overlapping set

$$\mathscr{K}_1 \cap \mathscr{K}_2 = \{x\}$$

contains only one point. The structure $\mathscr{L} = (\mathscr{K}, \{1,2\}, \{\psi_1,\psi_2\})$ is finitely ramified and fulfills finite preimage property, where the injections ψ_1,ψ_2 are induced by shift σ_1,σ_2, that is, $\psi_a = \pi\circ\sigma_a\circ\pi^{-1}$ for $a\in\{1,2\}$.

Finitely ramified and finite preimage do not imply OSC.

Example (Continued)

On the other hand, any given finite word $\mathbf{x} \in S^*$ is a factor of $\mathbf{v}$, that is, $\mathbf{x} = \mathbf{v}|_{[m+1,m+n]} = \sigma^m(\mathbf{v})|_{[1,n]}$ for some $m, n \in \mathbb{N}$, where $\mathbf{v}|_{[m+1,m+n]} = v_{m+1} \dots v_{m+n}$ for $\mathbf{v} = v_1 v_2 \dots$. Since

$$\mathscr{P}_{\mathscr{L}} \supset \sigma^{m+1}(\mathscr{C}_{\mathscr{L}}) = \sigma^{m+1}(\pi^{-1}(\{x\})) = \sigma^{m+1}(\{2\mathbf{v}, 12^\infty\}) = \{\sigma^m(\mathbf{v}), 2^\infty\},$$

the intersection $\mathscr{P}_{\mathscr{L}} \cap [\mathbf{x}]$ is not empty, where the cylinder $[\mathbf{x}] = \sigma_{\mathbf{x}}(S^{\mathbb{N}})$. By the characterization of OSC, the structure $\mathscr{L}$ do not fulfills the open set condition.

OSC does not imply finite preimage.

Example (OSC $\not\Rightarrow$ finite preimage)

Define a equivalence relation $\sim$ on the admissible word set $S^{\mathbb{N}} = \{1,2,3\}^{\mathbb{N}}$ as follows. Two different words $\mathbf{w}, \mathbf{w}' \in E^{\mathbb{N}}$ are equivalent if they are of forms

$$\mathbf{w} = \mathbf{uv} \text{ and } \mathbf{w}' = \mathbf{uv}', \quad \mathbf{u} \in \{1,2,3\}^*, \mathbf{v}, \mathbf{v}' \in \{1,2\}^{\mathbb{N}}.$$

The complete metric space $\mathscr{K}$ is the quotient space $\{1,2,3\}^{\mathbb{N}}/\sim$ with quotient metric. Denote by π the quotient map. The structure $\mathscr{L} = (\mathscr{K}, \{1,2,3\}, \{\psi_1, \psi_2, \psi_3\})$ is finitely ramified with the injections ψ_a induced by shift σ_a, that is, $\psi_a = \pi \circ \sigma_a \circ \pi^{-1}$ for $a \in S$. In fact, the overlapping set

$$\mathscr{R}_{\mathscr{L}} = \mathscr{K}_1 \cap \mathscr{K}_2 = \{y\},$$

where y is the point $\pi(\{1,2\}^{\mathbb{N}})$. The preimage set $\pi^{-1}(y) = \{1,2\}^{\mathbb{N}}$ is infinite.

OSC does not imply finite preimage.

Example (Continued)

On the other hand, the post critical set

$$\mathscr{P}_{\mathscr{L}} = \bigcup_{n\geq 1} \sigma^n(\mathscr{C}_{\mathscr{L}}) = \bigcup_{n\geq 1} \sigma^n(\pi^{-1}(y)) = \{1,2\}^{\mathbb{N}}$$

is infinite. Certainly, $\mathscr{P}_{\mathscr{L}}$ is not dense in $S^{\mathbb{N}}$, and thus $\mathscr{L}$ fulfills the open set condition.

Implications in general situation.

In summary, we have

$$\text{p.c.f.} \Rightarrow \text{finitely ramified} + \text{open set ondition} + \text{finite preimage},$$

where each of the three conditions "finitely ramified", "open set condition" and "finite preimage" does not imply any of the others.

Doubling quotient metric.

We endow $S^{\mathbb{N}}$ with the metric

$$d(\mathbf{w},\mathbf{w}') = 2^{-|\mathbf{w}\wedge\mathbf{w}'|},$$

where $\mathbf{w}\wedge\mathbf{w}'$ is the longest common prefix of $\mathbf{w}$ and $\mathbf{w}'$. It is compatible with the product topology over $S^{\mathbb{N}}$ where $S^{\mathbb{N}}$ is considered as infinite product of discrete set S. The topology on $\mathscr{K}$ is always the same with the quotient topology induced by natural map π from $S^{\mathbb{N}}$ to $\mathscr{K}$. In this section, we focus on the quotient metric on $\mathscr{K}$ which is regarded as the intrinsic metric on $\mathscr{K}$.

Finitely ramified implies finite preimage and OSC within doubling metric.

Theorem (Ni and Wen)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure. If $\mathscr{K}$ is doubling with respect to the quotient metric, then finitely ramified implies finite preimage.

Theorem (Ni and Wen)

Let $\mathscr{L} = (\mathscr{K}, S, \{\psi_a\}_{a\in S})$ be a self-similar structure. If $\mathscr{K}$ is doubling with respect to the quotient metric, then $\mathscr{L}$ is finitely ramified implies that $\mathscr{L}$ fulfills open set condition.

In short, with metric space $\mathscr{K}$ doubling, we have

$$\text{p.c.f.} \Rightarrow \text{finitely ramified} \Rightarrow \text{open set condition} + \text{finite preimage}.$$

Summary

- In general case, we have

$$\text{p.c.f.} \Rightarrow \text{finitely ramified} + \text{open set ondition} + \text{finite preimage},$$

  where each of the three conditions "finitely ramified", "open set condition" and "finite preimage" does not imply any of the others.
- In doubling case, we have

$$\text{p.c.f.} \Rightarrow \text{finitely ramified} \Rightarrow \text{open set condition} + \text{finite preimage}.$$

- Kigami defined two concepts "minimal" and "Bernoulli self-similar measure" for self-similar structure and deduced that

$$\text{finite preimage} \Rightarrow \text{Bernoulli self-similar measure} \Rightarrow \text{minimal}.$$

  The relations between these two conditions and "OSC" or "finitely ramified" are **not known**.

Thank you.

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