Boundary theory and self-similar set

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Background

- M.Denker and H.Sato 2001 introduced, on the finite word space Σ^* of the Sierpinski gasket (SG), a natural transition probability resembles the IFS construction that the chain goes to its descendants as well as to those of the neighboring states. They showed that the Martin boundary $\mathcal{M}$ is homeomorphic to SG, and the hitting probability on $\mathcal{M}$ is the canonical Hausdorff measure. Moreover, they identified a subclass of "strongly harmonic functions" on the Martin boundary that coincides with Kigami's harmonic functions. The above consideration has been extended to the class of *mono-cyclic* post critically finite (p.c.f.) self-similar sets by the authors (H.B. Ju, K.S. Lau and X.Y. Wang, Tran. Amer. Math. Soc. 2010).

Background

- Kaimanovich (2003) introduced a hyperbolic graph structure ("augmented" tree) on the symbolic space of the Sierpinski gasket, i.e., by adding a set of horizontal edges on the tree that is determined by the intersections of the cells, and showed that the gasket can be identified naturally as the boundary of the hyperbolic graph.
- Ancona (1978) showed that on a hyperbolic graph, the Martin boundary agrees with the hyperbolic boundary under the assumption of uniformly "irreducible" random walk with bounded range and spectral radius < 1.

Hyperbolic Graph

Let $(G, \mathcal{G})$ be a graph with countably infinite vertices set G. We assume that the graph is connected and "no loop". A graph carries an integer-valued metric $d(x, y)$, which is the minimal length of all paths from x to y.

For $x \in G$, we call

$$\deg(x) = \{y \in G : \ (x, y) \in \mathcal{G}\}$$

the *degree* of x.

We say a graph is *local finite* if there exists a constant $c > 0$ such that $\max\{\deg(x) : \ x \in G\} \leq c$.

We fix a reference point $o \in G$ and call it the *root*. Denote $|x| = d(o, x)$.

Recall that the *Gromov product* of two vertices $x, y \in G$ is defined by

$$|x \wedge y| = \frac{1}{2}(|x| + |y| - d(x, y)). \tag{1}$$

Definition

We say a graph $(G, \mathcal{G})$ is δ-hyperbolic (with respect to the root o) if there exists a constant $\delta > 0$ such that

$$|x \wedge y| \geq \min\{|x \wedge z|,\ |z \wedge y|\} - \delta, \quad \forall x, y, z \in G. \tag{2}$$

As in Woess's book, we choose $a > 0$ such that $a' = e^{\delta a} - 1 < \sqrt{2} - 1$, where δ is as in (2). Define for $x, y \in G$,

$$\rho_a(x,y) = \begin{cases} \exp(-a|x \wedge y|), & x \neq y, \\ 0, & x = y. \end{cases} \tag{3}$$

Then

$$\rho_a(x,y) \leq (1 + a') \max\{\rho_a(x,z), \rho_a(y,z)\}, \quad \forall\, x, y, z \in G. \tag{4}$$

This means $\rho_a(\cdot,\cdot)$ is an *ultra-metric*. It is not a metric, but is equivalent to the following metric:

$$\theta_a(x,y) = \inf\{\sum_{i=1}^{n} \rho_a(x_{i-1}, x_i) : \ n \geq 1,\ x = x_0, x_1, \ldots, x_n = y,\ x_i \in G\},$$

in sense that $(1 - 2a')\rho_a \leq \theta_a \leq \rho_a$

$\{x_n\}$ is Cauchy in the ultra-metric $\rho_a(x, y)$ if and only if $\lim_{m,n\to\infty} |x_m \wedge x_n| = \infty$.

Definition

Let $\widehat{G}$ denote the completion of the graph G under ρ_a. We call $\partial G = \widehat{G} \setminus G$ the hyperbolic boundary of G.

The hyperbolic boundary ∂G is a compact set.

Induced Graphs by IFS

Let $\{S_j\}_{j=1}^N$ be an IFS of similitudes on $\mathbb{R}^d$. For convenience, we assume that all the contraction ratios are equal. We say $\mathbf{i}, \mathbf{j} \in \Sigma^* := \cup_{n=0}^{\infty}\{1, 2, \cdots, N\}^n$ are *equivalent* and denote by $\mathbf{i} \sim \mathbf{j}$ if and only if $S_{\mathbf{i}} = S_{\mathbf{j}}$. It is clear that $\sim$ defines an equivalence relation. We denote by X the quotient space $\Sigma^*/\sim$, and $[\mathbf{i}]$ the equivalence class of $\mathbf{i}$. For $x = [\mathbf{i}] \in X$,

There is a natural graph on X: For $x = \{\mathbf{i}_1, \cdots, \mathbf{i}_n\}$, $y = \{\mathbf{j}_1, \cdots, \mathbf{j}_m\} \in X$, we say that there is an edge between x and y if $|x| = |y| - 1$ and $\mathbf{j}_k = \mathbf{i}_\ell i$ for some i, k, ℓ. We denote by $\mathcal{E}_v$ the above edge set. For $y \in X$, we use the notation y^{-1} to denote any one of $x \in X$ such that $(x, y) \in \mathcal{E}_v$ and $|y| = |x| + 1$. More general, define inductively $y^{-n} = (y^{-(n-1)})^{-1}$. Also by abusing notation, we write $(\mathbf{i}, \mathbf{j}) \in \mathcal{E}_v$ to mean that $([\mathbf{i}], [\mathbf{j}]) \in \mathcal{E}_v$.

In order to describe the self-similar set K, we need more edges. Let

$$\mathcal{E}_v^+ = \{(x,y): \ |y| = |x|+1, \ S_x(K)\cap S_y(K) \neq \emptyset, \ x \neq y^{-1}\};$$

and let

$$\mathcal{E}_h = \{(x,y): \ |y| = |x|, \ S_x(K)\cap S_y(K) \neq \emptyset\}.$$

We call an edge in $\mathcal{E}_v \cup \mathcal{E}_v^+$ a *vertical edge*, and an edge in $\mathcal{E}_h$ a *horizontal edge*. Let

$$\mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h, \quad \text{and} \quad \mathcal{E}^\diamond = \mathcal{E}_v \cup \mathcal{E}_v^+.$$

Example 1.

Let $S_i(x) = \frac{1}{2}(x+i)$, $x \in \mathbb{R}$, $i = 0, 1, 2$. $K = [0, 2]$.
$\{1,2,3\}^2/\sim = \{\{00\}, \{01\}, \{02, 10\}, \{11\}, \{12, 20\}, \{21\}, \{22\}\}$.
The vertex $\{02, 10\}$ have two ancestors $\{0\}$ and $\{1\}$.
$(0, 02), (1, 02) \in \mathcal{E}_v$ (abusing the notation).

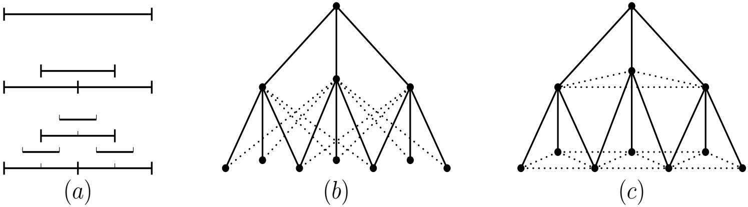


Figure: Example 1, (a) the iteration; (b) the graph $(X, \mathcal{E}^\diamond)$; (c) the graph $(X, \mathcal{E})$. The solid lines in $(b), (c)$ are edges in $\mathcal{E}_v$; the doted lines in $(b), (c)$ are edges in $\mathcal{E}_v^+$, and $\mathcal{E}_h$ respectively.

Definition

We say that the IFS $\{S_j\}_{j=1}^N$ satisfies the weak separation condition (WSC) if there exists some constant $\gamma > 0$ and a compact subset $D \subset \mathbb{R}^d$ with non-empty interior and $\cup_{j=1}^N S_j(D) \subset D$, such that for any $n \geq 1$ and $x \in \mathbb{R}^d$

$$\#\{S \in \mathcal{A}_n : \ x \in S(D)\} \leq \gamma,$$

where $\mathcal{A}_n = \{S_x : |x| = n, \quad x \in X\}$.

Theorem

Assume that the IFS satisfies the weak separation condition. Then the induced graphs $(X, \mathcal{E})$ and $(X, \mathcal{E}^\diamond)$ are local finite.

Proposition

Suppose the IFS satisfies the WSC, or the self-similar set K has positive Lebesgue measure. Then graphs $(X, \mathcal{E})$ and $(X, \mathcal{E}^\diamond)$ are hyperbolic. The hyperbolic boundaries $\partial X = \partial X^\diamond$, and the hyperbolic metrics ρ_a and $\rho_a^\diamond$ are equivalent.

Theorem

There is a map $\Phi : \ \partial X \longrightarrow K$ is a bijection and there exists a constant $C > 0$ such that

$$|\Phi(\xi) - \Phi(\eta)| \leq C\rho_a(\xi, \eta)^\alpha, \quad \forall \xi, \eta \in \partial X, \tag{5}$$

where $\alpha = -\log r/a$. In particular ∂X is homeomorphic to the self-similar set K.

(H) There exists a constant $C' > 0$ such that for any integer $n > 0$ and $\mathbf{u}, \mathbf{v} \in X_n := \{x \in X : \ |x| = n\}$, either

$$S_{\mathbf{u}}(K) \cap S_{\mathbf{v}}(K) \neq \emptyset \quad \text{or} \quad |S_{\mathbf{u}}(x) - S_{\mathbf{v}}(y)| \geq C' r^n, \quad \forall x, y \in K.$$

Proposition

Suppose the IFS $\{S_j\}_{j=1}^N$ in above theorem satisfies in addition condition (H). Then there exists a constant $C > 0$ such that for any $\xi, \eta \in \partial X$,

$$C^{-1}|\Phi(\xi) - \Phi(\eta)| \leq \rho_a(\xi, \eta)^\alpha \leq C|\Phi(\xi) - \Phi(\eta)|, \tag{6}$$

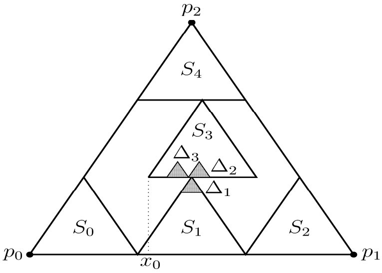
where $\alpha = -\log r/a$.

Example 2:

$S_i(x) = \frac{1}{3}(x + q_i)$, $q_i \in \mathbb{R}^2$, $i = 0, 1, \cdots, 4$ be the five maps. Let x_0 be such that

$$\frac{1}{2} - x_0 = \sum_{k=1}^{\infty} 3^{-n_k}, \quad n_k = 1 + \frac{k(k+1)}{2}, \; k = 1, 2, \cdots .$$

The condition (H) does not hold and the self-similar set K and the hyperbolic boundary ∂X are not Holder equivalent.



1. The above theorem extend the results in Lau and Wang 2009 (Indiana U. Math. J.) where the IFS satisfies the OSC.

2. Lau and Luo use the above to study the Lipschiz equivalency for the self-similar sets.

3. Moran sets as hyperbolic boundary was studied by Jun Jason Luo recently.

Questions

Question 1: Does that graph $(X, \mathcal{E})$ or $(X, \mathcal{E}^{\diamond})$ is local finite imply that the IFS satisfies the WSC?

Question 2: Are the conditions that the IFS satisfies the WSC or the self-similar set has positive Lebesgue measure necessary for the graphs to be hyperbolic?

Martin Boundary

Let $p(x, y)$ be a transition probability on in countable infinite set X. Assume it is transient and there exist some $\vartheta \in X$ s.t $p_n(\vartheta, x) > 0$ for any x and some n.

We define the Green function and the Martin kernel $g, \ k: \ X \times X \to \mathbb{R}$ by

$$g(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} p_n(\mathbf{x}, \mathbf{y}), \quad \text{and} \quad k(\mathbf{x}, \mathbf{y}) = \frac{g(\mathbf{x}, \mathbf{y})}{g(\vartheta, \mathbf{y})},$$

We define the Martin metric on X by

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{u} \in X} a(\mathbf{u}) |k(\mathbf{u}, \mathbf{x}) - k(\mathbf{u}, \mathbf{y})|. \tag{7}$$

where $a(\mathbf{u}) > 0$ is a positive function on X. Let $\hat{X}$ be the ρ-completion of X. We call $\partial X = \hat{X} \setminus X$ the Martin boundary.

Harmonic Function, Space of Exit

Let f be a non-negative function on X, we define the Markov operator

$$(Pf)(\mathbf{x}) = \sum_{\mathbf{y}\in X} p(\mathbf{x},\mathbf{y})f(\mathbf{y}), \quad \mathbf{x}\in X.$$

We say a function $h \geq 0$ on X is harmonic if $Ph = h$.

For any non-negative harmonic function h, there exists a measure μ_h on the Martin boundary $\mathcal{M}$ (which may not unique) such that

$$h(\cdot) = \int_{\mathcal{M}} k(\cdot, y)d\mu_h(y). \tag{8}$$

The set of all $y \in \mathcal{M}$ for which $\mu_{k_y} = \delta_y$ (the point mass measure at y) is called the space of exits, and is denoted by $\mathcal{M}_{\min}$.

Transition Probability

Let $p(\cdot,\cdot)$ be a transition probability on X, s.t.

(A1) $p(\mathbf{x},\mathbf{y})>0$ if $\mathbf{x}=\mathbf{y}^-$;

(A2) $p(\mathbf{x},\mathbf{y})>0$ implies that either $\mathbf{x}=\mathbf{y}^-$ or $(\mathbf{x},\mathbf{y}^-)\in\mathcal{E}$;

(A3) $p(\mathbf{x},\mathbf{y})>0$, $\forall\mathbf{y}$ s.t. $\mathbf{y}^-\asymp\mathbf{x}$ and $\mathbf{y}\asymp\mathbf{xk}$ for some $\mathbf{k}\in\Sigma^*$;

(A4) $\inf\{p(\mathbf{x},\mathbf{y})>0:\ \mathbf{x},\mathbf{y}\in X\}:=a>0$.

Transition Probability

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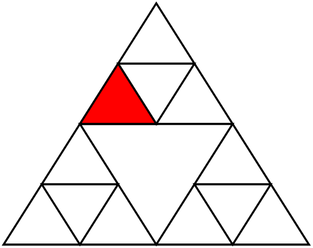
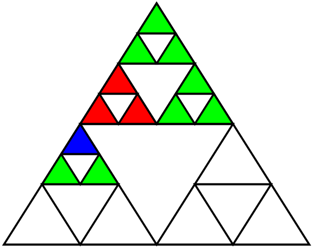
(A4) $\inf\{p(\mathbf{x},\mathbf{y}) > 0 : \ \mathbf{x},\mathbf{y} \in X\} := a > 0$.





DS-type transition probability

(A5) *There exists a constant* $C_0 > 0$ *such that*

$$\frac{g(\vartheta, \mathbf{y}_1)}{g(\vartheta, \mathbf{y}_2)} \leq C_0, \tag{9}$$

for any $\mathbf{x} \in \Sigma^\infty$ *and all* $\mathbf{y}_1, \mathbf{y}_2 \in \mathcal{A}(\mathbf{x})$ *with* $|\mathbf{y}_1| = |\mathbf{y}_2|$. where

$$\mathcal{A}(\mathbf{x}) = \{\mathbf{y} \in X : \; p(\mathbf{y}, \mathbf{x}|_n) > 0 \text{ for some } n\}.$$

Definition: We call a Markov chain on the symbolic space X a DS-type if the transition probability satisfies assumptions (A1)-(A5), Let $\{S_j\}_{j=1}^N$ be similitudes on $\mathbb{R}^d$ with the same contraction ratio r, and satisfies the open set condition (OSC). Let $\#[\mathbf{x}]$ denote the number of $\mathbf{y}$ conjugated to $\mathbf{x}$ ($\mathbf{x} \asymp \mathbf{y}$, iff $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}_h$ and $\mathbf{x}^- \neq \mathbf{y}^-$) and let

$$M = \max\{\#[\mathbf{x}] : \ \mathbf{x} \in \Sigma^*\}(< \infty) \quad \text{and} \quad 0 < a \leq \frac{1}{(M+1)N}.$$

Define

$$p(\mathbf{x}, \mathbf{y}) = \begin{cases} a, & \text{if } \mathbf{x} \asymp \mathbf{y}^-; \\ \frac{1}{N} - \#[\mathbf{x}]a, & \text{if } \mathbf{x} = \mathbf{y}^-; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{S_j\}_{j=1}^N$ be similitudes on $\mathbb{R}^d$ with the same contraction ratio r, and satisfies the open set condition (OSC). Let $\#[\mathbf{x}]$ denote the number of $\mathbf{y}$ conjugated to $\mathbf{x}$ ($\mathbf{x} \asymp \mathbf{y}$, iff $(\mathbf{x}, \mathbf{y}) \in \mathcal{E}_h$ and $\mathbf{x}^- \neq \mathbf{y}^-$) and let

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Proposition: For the above $p(\cdot,\cdot)$, we have $g(\vartheta, \mathbf{y}) = N^{-n}$ for $\mathbf{y} \in \Sigma^n$. Hence (A5) is satisfied.

Examples

Example 3.1. We can define a transition probability on the Sierpinski gasket. let $N[\mathbf{x}]$ denote the number of neighbors of $\mathbf{x}$, and let

$$p(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{1}{3(N[\mathbf{x}]+1)} , & \text{if } \mathbf{x} = \mathbf{y}^{-} \text{ or } \mathbf{x} \sim \mathbf{y}^{-}; \\ 0 , & \text{otherwise} . \end{cases}$$

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Example 3.2. We can use the same consideration on the Sierpinski carpet with $\mathbf{x}\sim\mathbf{y}$ if $\dim\big(S_{\mathbf{x}}(K)\cap S_{\mathbf{y}}(K)\big)=1$. There are 8 maps, each $\mathbf{x}$ has 1, 2 or no conjugate (hence $\#[\mathbf{x}]=0,1,\text{or } 2)$. Similar to Example 2.1, we can define a transition probability

$$p(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{1}{8(\#[\mathbf{x}]+1)}\ , & \text{if } \mathbf{x}=\mathbf{y}^{-} \text{ or } \mathbf{x}\asymp\mathbf{y}^{-};\\ 0\ , & \text{otherwise}\ . \end{cases} \tag{10}$$

The above examples satisfy (A1)-(A5).

Theorem (Lau and Wang)

Suppose the IFS satisfies the OSC, and the transition probability is of DS-type on the augmented tree $(X, \mathcal{E})$. Then the Martin boundary $\mathcal{M}$, the hyperbolic boundary $\partial_H X$ and the self-similar set K are homeomorphic. Moreover, the space of exits $\mathcal{M}_{\min}$ equals $\mathcal{M}$.

Theorem (Lau and Wang)

Let $\{S_j\}_{j=1}^N$ be an IFS of similitudes that satisfies the OSC and has equal contraction ratio r. Suppose the transition probabilty is of DS-type and the Green function satisfies $g(\vartheta, \mathbf{v}) = N^{-|\mathbf{v}|}$, $\forall\ \mathbf{v} \in \Sigma^*$. Then the spectral measure μ_1 is the normalized s-Hausdorff measure with $s = |\log N / \log r|$.

If we modify the definition of the Martin metric as

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{n=0}^{\infty} \left(\frac{\alpha}{N}\right)^n \max_{|\mathbf{u}|=n} \{|k(\mathbf{u}, \mathbf{x}) - k(\mathbf{u}, \mathbf{y})|\}, \tag{11}$$

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Theorem (Lau and Wang)

Assume the conditions in above Theorem hold, then $\exists C > 0$ *s.t.*

$$\frac{1}{C}\rho_H^s(\mathbf{x},\mathbf{y}) \le \rho(\mathbf{x},\mathbf{y}) \le C\rho_H^s(\mathbf{x},\mathbf{y})|\log \rho_H(\mathbf{x},\mathbf{y})|, \quad \forall \mathbf{x},\mathbf{y} \in \Sigma^\infty,$$

where $s = -\log \alpha / a$ *(the constant* a *is as in the definition of hyperbolic matric).*

$\mathcal{T}hankYou$