Localized Birkhoff average in beta dynamical systems

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Background

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Let $(X, T, \mathcal{B}, \mu)$ be a measure-theoretic dynamical system.

Birkhoff's ergodic Thm

- (I). For any $f \in \mathbb{L}^1(\mu)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \mathbb{E}(f|\mathcal{I})(x) \quad \mu - a.s.$$

  where $\mathcal{I}$ is the σ-algebra of T-invariant sets.

- (II). If, furthermore, T is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f(x) d\mu \quad \mu - a.s.$$

Multifractal analysis of Birkhoff average :

Classical case :

$$\left\{x : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \alpha\right\}$$

Remark : The Birkhoff average of a point x should depend on x itself.

Consider the size of the set

$$\left\{x : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \psi(x)\right\}$$

This is called as *localized Birkhoff average* : instead of a constant, the function ψ here varies with x.

Known Results

- Barral & Seuret 2011 : Localized Jarnik Thm :

$$\Big\{x : \delta(x) = f(x)\Big\}$$

where $\delta(x)$ is the exact diophantine exponent of x :

$$\delta(x) = \sup\Big\{\delta : |x - p/q| < q^{-\delta}, \text{ i.o. } q \in \mathbb{N}\Big\}.$$

- Barral & Qu 2012 : Localized multifractal analysis :

$$E_f(\psi) := \Big\{x : \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} f(T^j x) = \psi(x)\Big\}$$

where (X, T) is a subshift of finite type.

Theorem (Barral & Qu, 2012)

Let f, ψ be two continuous function.

$$\dim_H E_f(\psi) = \sup\left\{ \frac{h_\mu}{\int \log |T'| d\mu} : \mu \in \mathfrak{M}(T), \int f d\mu \in \mathfrak{D}(\psi) \cap \mathfrak{L}_f \right\}$$

where $\mathfrak{M}(T)$ denotes the collection of all T-inv. probability measures and h_μ is the measure theoretic entropy of μ.

Our concern

Question : Localized Birkhoff average in beta expansions

Intention :

- The beta-dynamical system for a general β is not usually a subshift of finite type with mixing properties.
- When can we solve the question in beta-dynamical system by applying the method that *approximating the system by subsystems of subshift of finite type ?*

Beta expansion

Notation

- Algorithm. For $\beta > 1$,

$$Tx = \beta x - \lceil \beta x \rceil + 1,$$

- Expansion. Every $x \in (0, 1]$ can be expressed uniquely as an infinite series

$$x = \frac{\epsilon_1(x,\beta)}{\beta} + \cdots + \frac{\epsilon_n(x,\beta)}{\beta^n} + \cdots$$

- Expansion of 1. Write

$$1 = \frac{\epsilon_1^*(\beta)}{\beta} + \cdots + \frac{\epsilon_n^*(\beta)}{\beta^n} + \cdots$$

- Parry number. Call β a Parry number, if

$$(\epsilon_1^*(\beta), \cdots, \epsilon_n^*(\beta), \cdots) \text{ eventually periodic.}$$

- Admissible sequence Σ_β : $(\epsilon_1, \cdots, \epsilon_n)$ is called admissible if $\exists x$ s.t. $\epsilon_j(x,\beta) = \epsilon_j$ for $1 \le j \le n$.

- Σ_β^n : β-admissible of length n ;
- Cylinder : for any $(w_1, \cdots, w_n) \in \Sigma_\beta^n$,

$$I_n(w) = \Big\{x : w_k(x, \beta) = w_k, 1 \le k \le n\Big\}.$$

Basic properties :

- Characterization on admissible sequence (W. Parry) : $(w_1, \cdots, w_n)$ is β-admissible if

$$w_k, \cdots, w_n \le w_1^*(\beta), \cdots, w_{n-k}^*(\beta), \quad 1 \le k \le n.$$

- Monotonicity : if $\beta < \beta'$

$$\Sigma_\beta^n \subset \Sigma_{\beta'}^n, \quad n \ge 1.$$

- Cardinality of Σ_β^n (A. Renyi) :

$$\beta^n \le \sharp\Sigma_\beta^n \le \beta^{n+1}/(\beta+1), \ n \ge 1.$$

A special feature : Lack of Markov properties

For any given $(\epsilon_1, \cdots, \epsilon_n) \in \Sigma_\beta^n$, the length of a cylinder

$$I_n(\epsilon_1, \cdots, \epsilon_n) := \{x \in [0,1] : \epsilon_k(x) = \epsilon_k, 1 \le k \le n\}$$

satisfies that

$$\mathbf{0} < \left| I(\epsilon_1, \cdots, \epsilon_n) \right| \le \beta^{-n}. \tag{1}$$

But when β is a Parry number, there is a universal constant C such that

$$C\beta^{-n} \le \left| I_n(\epsilon_1, \cdots, \epsilon_n) \right| \le \beta^{-n}.$$

Remark : For a general $\beta > 1$, the lower bound in (1) cannot be improved, , which constitutes one main difficulty in studying the metric theory related to beta expansions.

How to overcome this difficulty ?

One method

Approximation :

Step 1. Approximate β by Parry numbers from below : let β_M be the solution to the equation

$$1 = \frac{\epsilon_1^*(\beta)}{x} + \cdots + \frac{\epsilon_M^*(\beta)}{x^M}.$$

Then β_M is a Parry number and $\beta_M \le \beta$.

Step 2. Only concentrate on x such that the digits sequence

$$\big(\epsilon_1(x,\beta), \cdots, \epsilon_n(x,\beta), \cdots\big) \in \Sigma_{\beta_M}.$$

Then for any such x,

$$\beta^{-(n+M)} \le \big|I_n(x)\big| \le \beta^{-n}.$$

Question : how many are lost ?

Character on bad points

Lemma (Tan-W, 2011)

$w \in \Sigma_\beta \setminus \Sigma_{\beta_M}$ iff $(\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta))$ is a subword of w.

Define map π on Σ_β,

$$\begin{aligned} \pi : w &\to w^* \\ \text{change } \underbrace{(\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta))}_{\text{in } w} &\to (\epsilon_1^*(\beta), \cdots, \epsilon_M^*(\beta) - 1) \end{aligned}$$

Lemma (Tan-W, 2011)

For any $w \in \Sigma_\beta^n$, write $\pi(w) = w^*$. Then

$$w^* \in \Sigma_{\beta_M}^n, \quad \sharp\pi^{-1}(w^*) \leq 2^{n/M}.$$

Since the digits in w and w^* differ only at very rare positions,

$$\Big|\sum_{j=0}^{n-1} f(T^j x) - \sum_{j=0}^{n-1} f(T^j x^*)\Big| < n\epsilon$$

when $M \gg 1$ for any $x \in I_n(w)$, $x^* \in I_n(w^*)$.

An application of above lemmas.

- Let φ be a continuous function on $[0,1]$, then the pressure function

$$P(\varphi, T_\beta) = \lim_{\beta' \to \beta} P(\varphi, T_{\beta'}).$$

Results

Let f, ψ be two continuous functions. Write

$$\mathfrak{D}(\psi) = \{\psi(x) : x \in [0,1]\}, \quad \mathfrak{L}_f = \left\{ \int f d\mu : \mu \in \mathfrak{M}(T) \right\}$$

Define

$$E_f(\psi) := \left\{ x : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \psi(x) \right\}.$$

Theorem (Tan-W-Wu-Xu, 2012)

$$\dim_H E_\psi(f) = \sup \left\{ \frac{h_\mu}{\log \beta} : \mu \in \mathfrak{M}(T) \text{ and } \int \psi d\mu \in \mathfrak{D}(\psi) \cap \mathfrak{L}_f \right\},$$

where $\mathfrak{M}(T)$ denotes the collection of all T-invariant probability measures and h_μ is the measure theoretic entropy of μ.

Sketch of Proof

Just consider the classic case :

$$E_f(\alpha) := \left\{ x : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \alpha \right\}$$

Upper bound :

Let $\alpha \in \mathbb{R}$, f a continuous function, $\epsilon > 0$. Denote by

$$\mathcal{F}_n(\beta, \epsilon) = \left\{ w \in \Sigma_\beta^n : \exists x \in I_n(w) \text{ s.t. } \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) - \alpha \right| < \epsilon \right\}.$$

Then by an evident covering argument,

$$\dim_{\mathsf{H}} E_f(\alpha) \le \lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{\log \sharp \mathcal{F}_n(\beta, \epsilon)}{n}.$$

Lower bound : Define

$$\mathcal{F}_n(\beta_M, \epsilon) = \left\{ w \in \Sigma_{\beta_M}^n : \exists x \in I_n(w) \text{ s.t. } \left| \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) - \alpha \right| < \epsilon \right\}.$$

Claim :

For any $w \in \mathcal{F}_n(\beta, \epsilon)$, we have $w^* \in \mathcal{F}_n(\beta_M, 2\epsilon)$.

Lemma

$$\sharp\mathcal{F}_n(\beta_M, \epsilon_n) \le \sharp\mathcal{F}_n(\beta, \epsilon) \le \sharp\mathcal{F}_n(\beta_M, 2\epsilon) \cdot 2^{n/M}.$$

Define a Cantor subset as

$$C_M := \bigcap_{n=1}^{\infty} \bigcup_{I_n \in \mathcal{F}_n(\beta_M, \epsilon_n)} I_n$$

Simple observations :

- For every $x \in C_M$, the lengths of the cylinders containing x are quite regular.
- For each level of the Cantor set C_M, the number of cylinders in $\mathcal{F}_n(\beta_M, \epsilon_n)$ is sufficiently large compared with $\mathcal{F}_n(\beta, \epsilon_n)$.

Thanks for your attention !