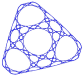
on **recent progress** in

spectral and vector analysis on fractafolds



Alexander Teplyaev
University of Connecticut

Hong Kong
December 2012

plan/wish list:

- **Motivation**

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  - **mathematical physics: the spectral dimension of the universe**

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- **1. Spectral analysis on finitely ramified symmetric fractafolds [Strichartz at al]**

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- **2. Differential forms and distributions on fractafolds [Rogers, Strichartz at al]**

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- **1. Spectral analysis on finitely ramified symmetric fractafolds [Strichartz at al]**
- **2. Differential forms and distributions on fractafolds [Rogers, Strichartz at al]**
- **3. Vector analysis for Dirichlet forms [Hinz at al]**

All papers are available in arXiv

mathematical physics: the spectral dimension of the universe

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]*The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark*

[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*

[3]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands*

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be “self-renormalizing” at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier meaother hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \rightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

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Abstract

The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered

Numerics for the Strichartz Hexacarpet

M. Begue, D. J. Kelleher, A. Nelson, H. Panzo, R. Pellico and A. Teplyaev, Random walks on barycentric subdivisions and Strichartz hexacarpet, arXiv:1106.5567 Experimental Mathematics, 21(4):402417, 2012

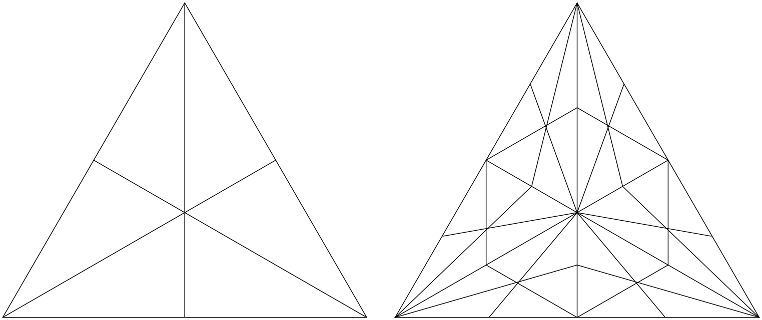
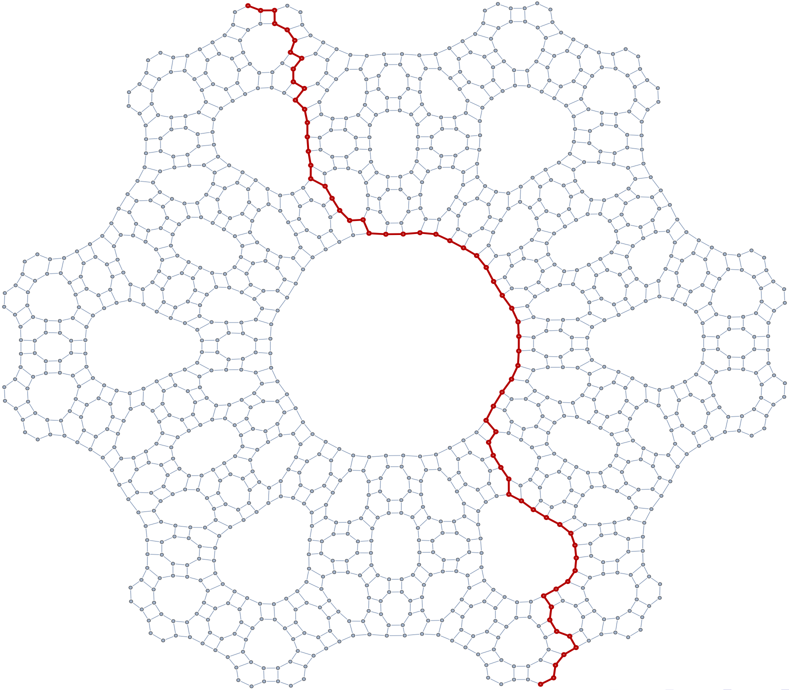
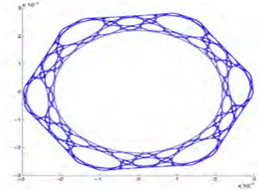
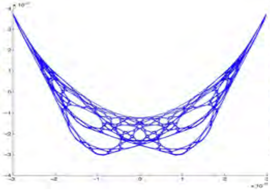


FIGURE 2.1. Barycentric subdivision

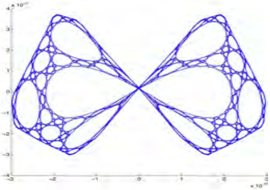




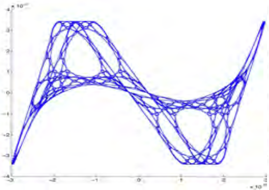
(A) (φ_2, φ_3)



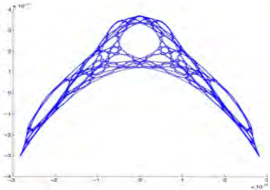
(B) (φ_2, φ_4)



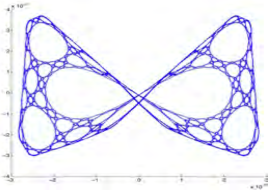
(C) (φ_2, φ_5)



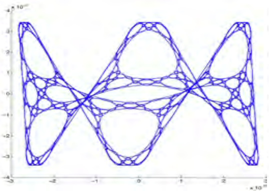
(D) (φ_2, φ_6)



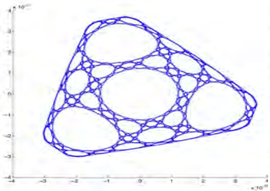
(E) (φ_3, φ_4)



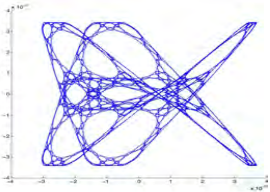
(F) (φ_3, φ_5)



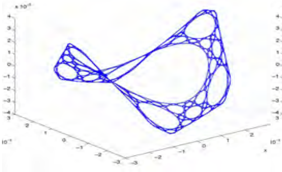
(G) (φ_3, φ_6)



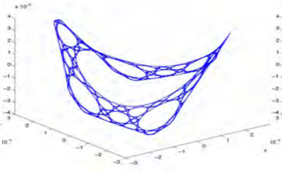
(H) (φ_4, φ_5)



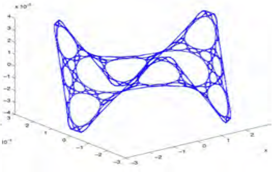
(I) (φ_4, φ_6)



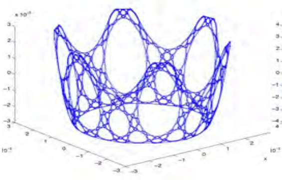
(A) $(\varphi_2, \varphi_3, \varphi_4)$



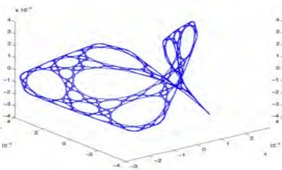
(B) $(\varphi_2, \varphi_3, \varphi_5)$



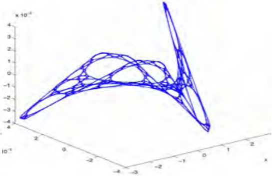
(C) $(\varphi_2, \varphi_3, \varphi_6)$



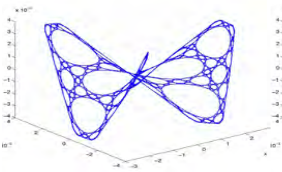
(D) $(\varphi_2, \varphi_3, \varphi_7)$



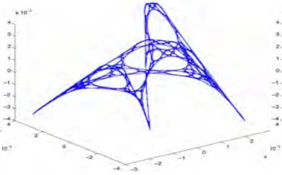
(E) $(\varphi_2, \varphi_4, \varphi_5)$



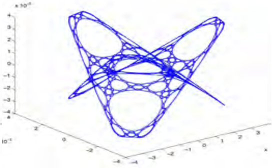
(F) $(\varphi_2, \varphi_4, \varphi_6)$



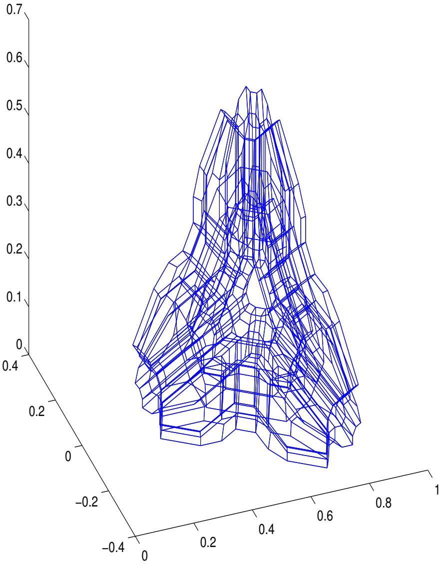
(G) $(\varphi_2, \varphi_5, \varphi_6)$



(H) $(\varphi_3, \varphi_5, \varphi_6)$



(I) $(\varphi_4, \varphi_5, \varphi_6)$


0.7
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0
0.2
0.4
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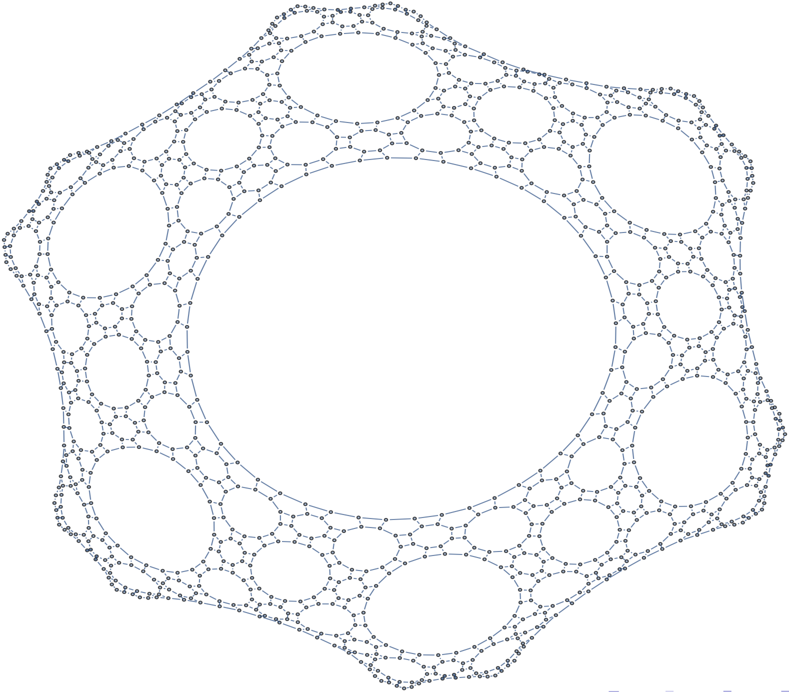
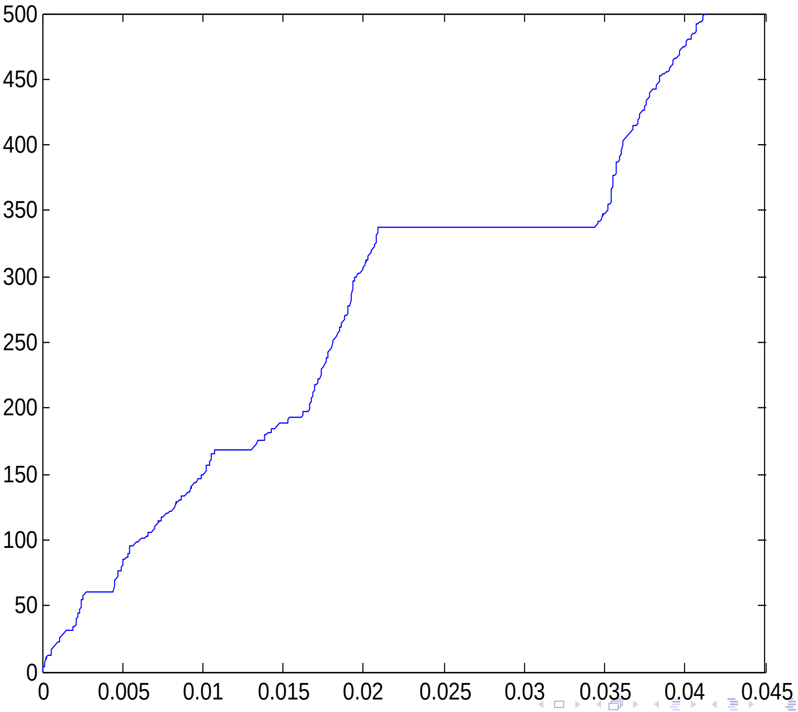
0.8
1





500
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50
0
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
0.04
0.045

λ_j	$n = 7$	$n = 8$
1	0.0000	0.0000
2	1.0000	1.0000
3	1.0000	1.0000
4	3.2798	3.2798
5	3.2798	3.2798
6	5.2033	5.2032
7	7.8389	7.8386
8	7.8389	7.8386
9	8.9141	8.9139
10	8.9141	8.9139
11	9.4951	9.4950
12	9.4952	9.4950
13	17.5332	17.5326
14	17.5332	17.5327
15	17.6373	17.6366
16	17.6373	17.6366
17	19.8610	19.8607
18	21.7893	21.7882
19	25.7111	25.7089
20	25.7112	25.7091

Table: Hexacarpet renormalized eigenvalues at levels $\mathbf{n = 7}$ and $\mathbf{n = 8}$.

	Level **n**						
c	1	2	3	4	5	6	7
1							
2	1.2801	1.3086	1.3085	1.3069	1.3067	1.3065	1.3064
3	1.2801	1.3086	1.3079	1.3075	1.3066	1.3065	1.3064
4	1.1761	1.3011	1.3105	1.3064	1.3068	1.3065	1.3065
5	1.1761	1.3011	1.3089	1.3074	1.3073	1.3065	1.3065
6	1.0146	1.2732	1.3098	1.3015	1.3067	1.3065	1.3064
7		1.2801	1.3114	1.3055	1.3071	1.3066	1.3065
8		1.2801	1.3079	1.3086	1.3075	1.3067	1.3065
9		1.2542	1.3191	1.2929	1.3056	1.3065	1.3065
10		1.2542	1.3017	1.3089	1.3069	1.3066	1.3065
11		1.2461	1.3051	1.3063	1.3048	1.3065	1.3065
12		1.2461	1.3019	1.3075	1.3068	1.3066	1.3065
13		1.1969	1.6014	1.0590	1.3068	1.3066	1.3065
14		1.1969	1.2972	1.3063	1.3078	1.3066	1.3065
15		1.2026	1.3059	1.3020	1.3060	1.3066	1.3065
16		1.2026	1.2993	1.3074	1.3071	1.3067	1.3065
17		1.1640	1.3655	1.2349	1.3064	1.3066	1.3065
18		1.1755	1.4128	1.2009	1.3069	1.3067	1.3065
19		1.1761	1.5252	1.1171	1.3073	1.3068	1.3066
20		1.1761	1.2988	1.3114	1.3077	1.3068	1.3065

Table: Hexacarpet estimates for resistance coefficient **c** given by $\frac{1}{6}\frac{\lambda_j^n}{\lambda_j^{n+1}}$.

Conjecture

We conjecture that

1. *on the Strichartz hexacarpet there exists a unique self-similar local regular conservative Dirichlet form* $\mathcal{E}$ *with resistance scaling factor* $\rho \approx 1.304$ *and the Laplacian scaling factor* $\tau = 6\rho$;
2. *the simple random walks on the repeated barycentric subdivisions of a triangle, with the time renormalized by* τ^n, *converge to the diffusion process, which is the continuous symmetric strong Markov process corresponding to the Dirichlet form* $\mathcal{E}$;
3. *this diffusion process satisfies the sub-Gaussian heat kernel estimates and elliptic and parabolic Harnack inequalities, possibly with logarithmic corrections, corresponding to the Hausdorff dimension* $\frac{\log(6)}{\log(2)} \approx 2.58$ *and the spectral dimension* $2\frac{\log(6)}{\log(\tau)} \approx 1.74$;
4. *the spectrum of the Laplacian has spectral gaps in the sense of Strichartz;*
5. *the spectral zeta function has a meromorphic continuation to* $\mathbb{C}$.

Historical perspective

triangles U_0, U_1, U_2, situés parallèlement à U, dont les intérieurs seront

Fig. 1.

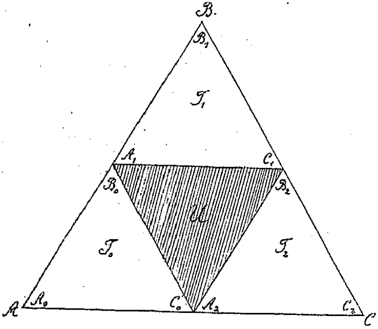
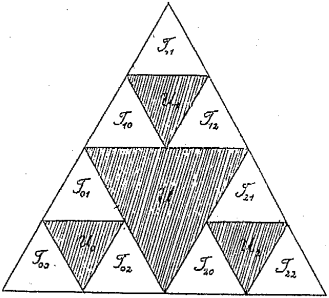


Fig. 2.



exclus (*fig.* 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi

ANALYSE MATHÉMATIQUE. — *Sur une courbe dont tout point est un point de ramification*. Note [1] de M. **W. Sierpinski**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe $\mathcal{C}$ un point p de cette courbe, s'il existe trois continus, sous-ensembles de $\mathcal{C}$, ayant deux à deux le point p et seulement ce point commun.)

Soient T un triangle régulier donné; A, B, C respectivement ses sommets: gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (*fig.* 1), dont trois, T_0, T_1, T_2, contenant respectivement les sommets A, B, C, sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles T_0, T_1, T_2 nous les désignerons respectivement:

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[1] Séance du 1er février 1915.

Fig. 3.

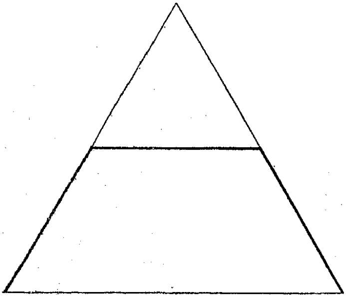
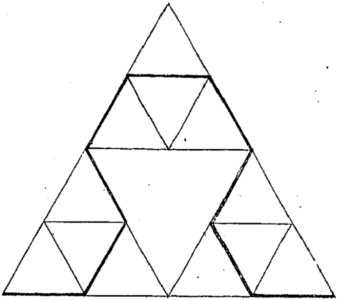


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\mathcal{C}$.

Donc, tous les points de la courbe $\mathcal{C}$, sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

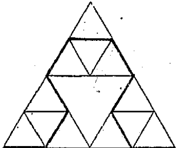


Fig. 6.

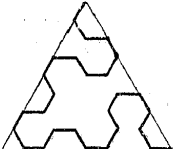


Fig. 5.

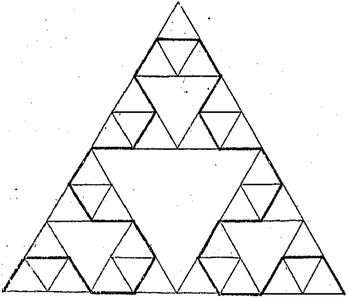
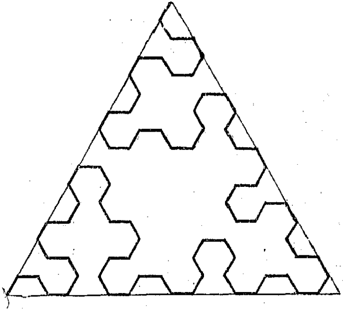


Fig. 6.



Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk[1], Zoran Šunić

Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA

Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

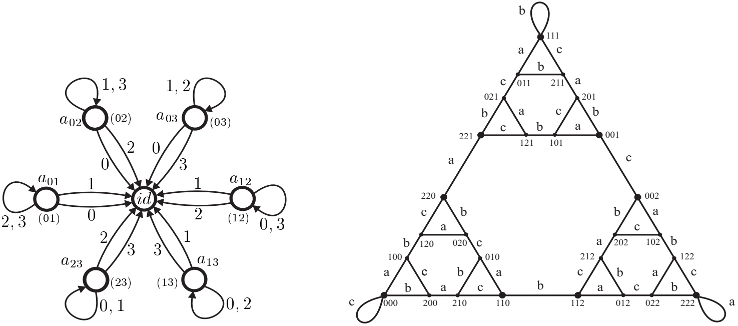
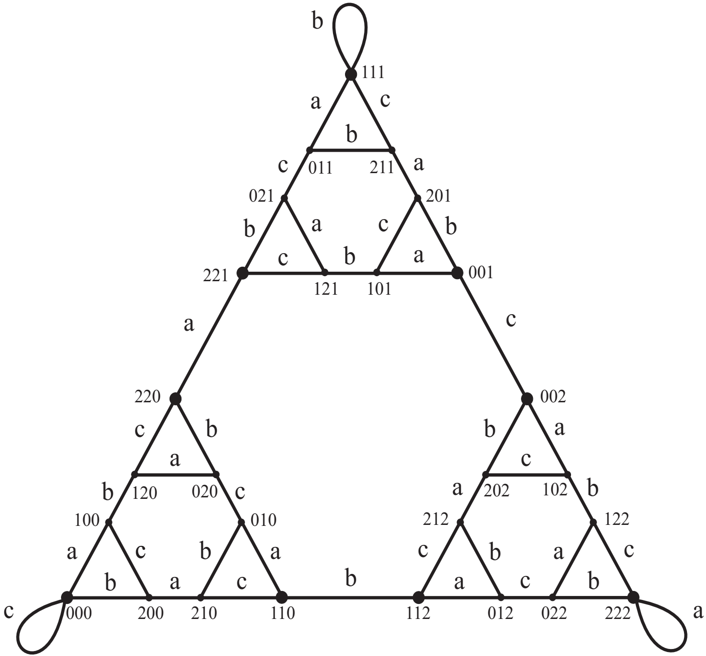


Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3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Early (physics) results on spectral analysis on fractals

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Early results on diffusions on fractals

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time $\mathbf{n}$ is at $\mathbf{X(n)}$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension $\mathbf{d}$ we show that, as $\mathbf{L} \to \infty$, the process $\mathbf{Y_L(t) \equiv X([(d+3)^L t])/2^L}$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|\mathbf{X(n)}| \sim \mathbf{n}^{\gamma},$$

where $\gamma = (\ln 2)/\ln(\mathbf{d}+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
- M. T. Barlow, R. F. Bass, *The construction of Brownian motion on the Sierpiński carpet.* Ann. Inst. Poincaré Probab. Statist. (1989)
- S. Kusuoka, *Dirichlet forms on fractals and products of random matrices.* (1989)
- T. Lindstrøm, *Brownian motion on nested fractals.* Mem. Amer. Math. Soc. **420**, 1989.
- J. Kigami, *A harmonic calculus on the Sierpiński spaces.* (1989)
- J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988) Cambridge Univ. Press, 1992.
- M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket.* (1992)
- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
- J. Kigami and M. L. Lapidus, *Weyl's problem for the spectral distribution of Laplacians on p.c.f. self-similar fractals.* Comm. Math. Phys. **158** (1993)

Main classes of fractals considered

- $\mathbf{[0, 1]}$

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- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

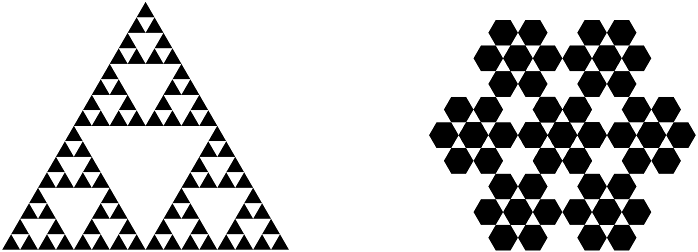

Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

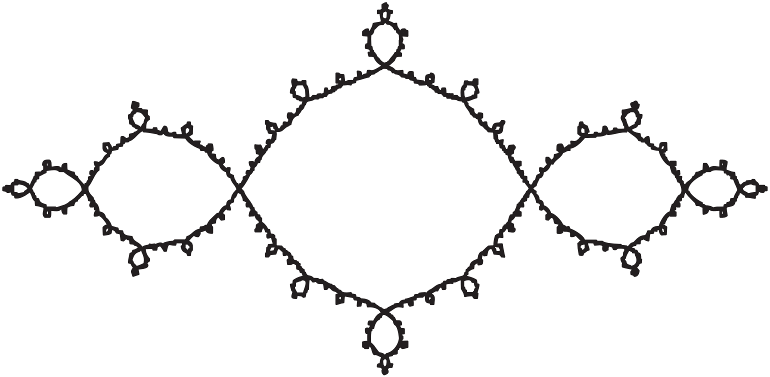

Figure: The basilica Julia set, the Julia set of $\mathbf{z^2 - 1}$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al., Rogers-T.).

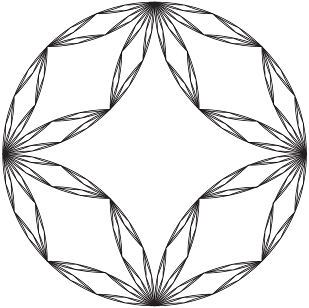

Figure: Diamond fractals, non-p.c.f., but finitely ramified

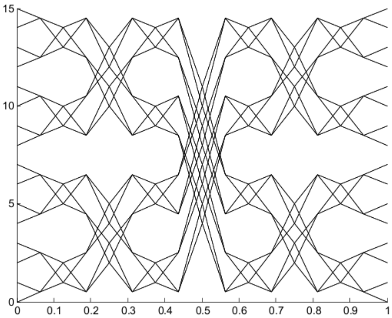

Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

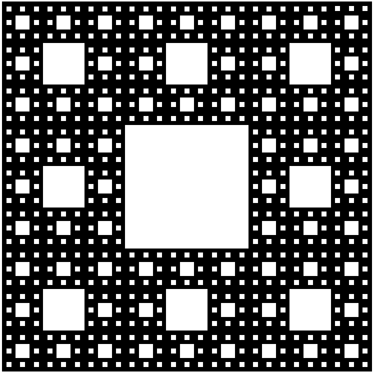

Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

$$\text{distance} \sim \sqrt{\text{time}}$$

"Einstein space–time relation for Brownian motion"

De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;

Li-Yau (1986) type estimates on a geodesically complete Riemannian manifold with **Ricci** $\geqslant$ **0**:

$$\mathbf{p_t(x, y) \sim \frac{1}{V(x, \sqrt{t})} \exp\left(-c\frac{d(x, y)^2}{t}\right)}$$

$$\mathbf{distance \sim \sqrt{time}}$$

Brownian motion on $\mathbb{R}^{\mathbf{d}}$: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{ct^{1/2}}$.

Anomalous diffusion: $\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| = \mathbf{o(t^{1/2})}$, or (in regular enough situations),

$$\mathbb{E}|\mathbf{X_t} - \mathbf{X_0}| \approx \mathbf{t^{1/d_w}}$$

with $\mathbf{d_w} > \mathbf{2}$.

Here $\mathbf{d_w}$ is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$\mathbf{p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)}$$

$$\textbf{distance} \sim \textbf{(time)}^{\frac{1}{d_w}}$$

$$\mathbf{d_H} = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = \mathbf{d_w} = \text{"walk dimension" } (\gamma\text{=diffusion index})$$

$$\frac{2d_H}{d_w} = \mathbf{d_S} = \text{"spectral dimension" (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian **heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of generalized Sierpiński carpets (includes cubes in $\mathbb{R}^{\mathbf{d}}$) there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties:

If it is not a cube in $\mathbb{R}^{\mathbf{n}}$, then

- $\mathbf{d_S} < \mathbf{d_H}$, $\mathbf{d_w} > \mathbf{2}$
- the energy measure and the Hausdorff measure are mutually singular;

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- Lipschitz functions are not of finite energy;
- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

Theorem. (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

- **(VD)**, **(EHI)** and **(RES)**
- **(VD)**, **(EHI)** and **(ETE)**
- **(PHI)**
- **(HKE)**

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

More on motivations and connections to other areas: Cheeger, Heinonen, Koskela, Shanmugalingam, Tyson

J. Cheeger, *Differentiability of Lipschitz functions on metric measure spaces,* Geom. Funct. Anal. **9** (1999) J. Heinonen, *Lectures on analysis on metric spaces.* Universitext. Springer-Verlag, New York, 2001. J. Heinonen, *Nonsmooth calculus,* Bull. Amer. Math. Soc. (N.S.) **44** (2007) J. Heinonen, P. Koskela, N. Shanmugalingam, J. Tyson, *Sobolev classes of Banach space-valued functions and quasiconformal mappings*. J. Anal. Math. 85 (2001)

In this paper the authors give a definition for the class of Sobolev functions from a metric measure space into a Banach space. They characterize Sobolev classes and study the absolute continuity in measure of Sobolev mappings in the "borderline case". Specifically, the authors prove that the validity of a Poincaré inequality for mappings of a metric space is independent of the target Banach space; they obtain embedding theorems and **Lipschitz approximation** of Sobolev functions; they also prove that pseudomonotone Sobolev mappings in the "borderline case" are absolutely continuous in measure, which is a generalization of the existing results by Y. G. Reshetnyak [Sibirsk. Mat. Zh. 28 (1987)] and by J. Malý and O. Martio [J. Reine Angew. Math. 458 (1995)]. The authors show that quasisymmetric homeomorphisms belong to a Sobolev space of

Remark: what are dimensions of the Sierpiński gasket?

- $\frac{\log 3}{\log \frac{5}{3}} \approx \mathbf{2.15} =$ Hausdorff dimension in effective resistance metric
- $\mathbf{2} =$ geometric, linear dimension
- $\frac{\log 3}{\log 2} \approx \mathbf{1.58} =$ usual Hausdorff (Minkowsky, box, self-similarity) dimension in Euclidean coordinates (geodesic metric)
- $\frac{2 \log 3}{\log 5} \approx \mathbf{1.37} =$ usual spectral dimension
- **...** **...** **...** $=$
  there are several Lyapunov exponent type dimensions related to harmonic functions and harmonic coordinates (Kajino, Ionescu-Rogers-T)
- $\mathbf{1} =$ topological dimension, martingale dimension
- $\frac{2 \log 2}{\log 5} \approx \mathbf{0.86} =$ polynomial spectral co-dimension (Grabner)?

still open math problems on fractals (with some progress made)

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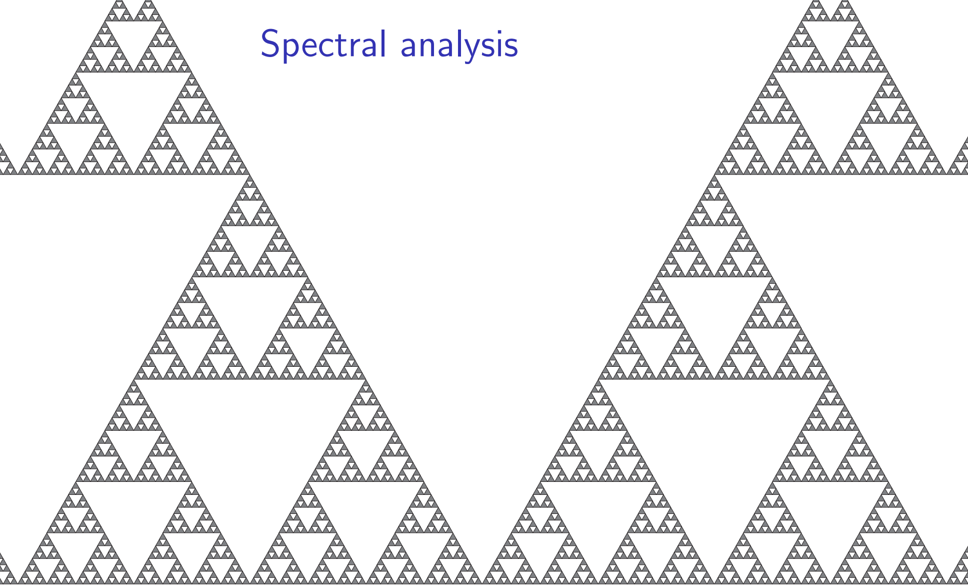
- PDEs involving derivatives, such as the Navier-Stokes equation?
- Derivatives on fractals; differential geometry of fractals (Rogers-Ionescu-T, Cipriani-Guido-Isola-Sauvageot, Hinz-Röcknert-T)?

Spectral analysis



A part of an infinite Sierpiński gasket.

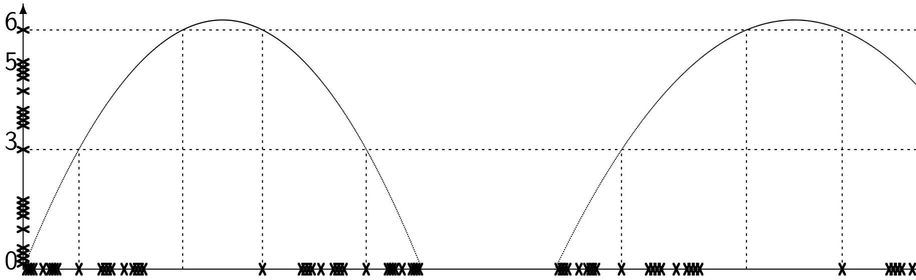
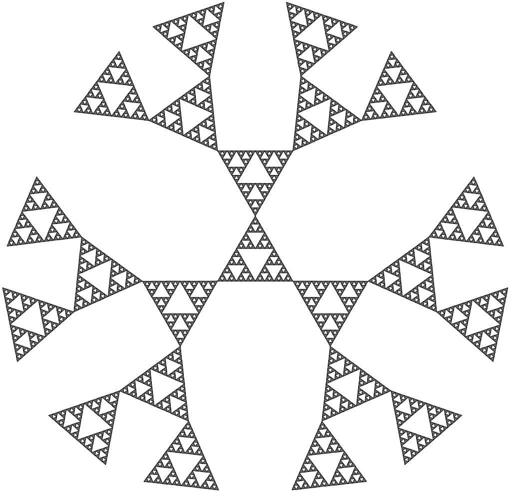
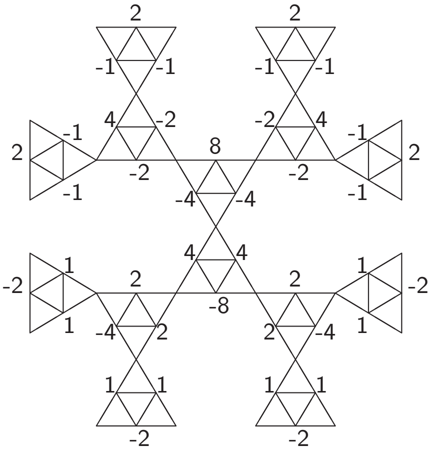


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

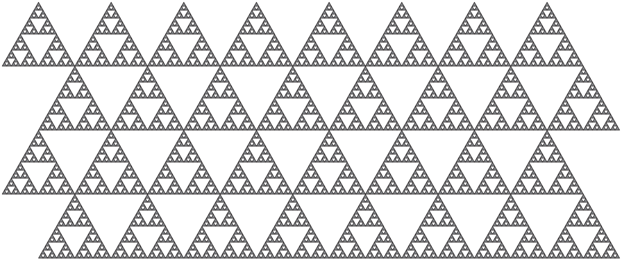
Theorem. (Béllissard 1988, T. 1998, Quint 2009)
On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** and a **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

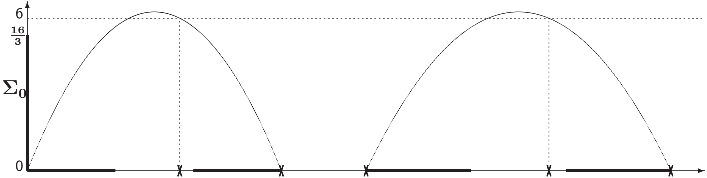


The Tree Fractafold.



An eigenfunction on the Tree Fractafold.





Theorem. (Strichartz, T. 2010) The Laplacian on the periodic triangular lattice finitely ramified Sierpiński fractal field consists of absolutely continuous spectrum and pure point spectrum. The **absolutely continuous spectrum is $\mathfrak{R}^{-1}[0, \frac{16}{3}]$.** The **pure point spectrum** consists of two infinite series of eigenvalues of infinite multiplicity. The spectral resolution is given in the main theorem.

Recent results on spectral analysis and applications

G. Derfel, P. J. Grabner, and F. Vogl: Laplace Operators on Fractals and Related Functional Equations, J. Phys. A, 45(46), (2012), 463001

N. Kajino, *Spectral asymptotics for Laplacians on self-similar sets*. J. Funct. Anal. 258 (2010)

N. Kajino, T, *Spectral gap sequence and on-diagonal oscillation of heat kernels*

Joe Chen, R. Strichartz, *Spectral asymptotics and heat kernels on three-dimensional fractal sponges*

J. F.-C. Chan, S.-M. Ngai, T, *One-dimensional wave equations defined by fractal Laplacians*

U. Freiberg, L. Rogers, T, *Eigenvalue convergence of second order operators on the real line*

B. Steinhurst, T, *Spectral Analysis and Dirichlet Forms on Barlow-Evans Fractals* arXiv:1204.5207

Derivations and Dirichlet forms on fractals

M. Ionescu, L. G. Rogers, A. Teplyaev, Derivations and Dirichlet forms on fractals, arXiv:1106.1450, Journal of Functional Analysis, 263 (8), p.2141-2169, Oct 2012

We study **derivations** and **Fredholm modules** on metric spaces with a local regular conservative Dirichlet form. In particular, on finitely ramified fractals, we show that there is a **non-trivial Fredholm module if and only if the fractal is not a tree** (i.e. not simply connected). This result relates Fredholm modules and topology, and refines and improves known results on p.c.f. fractals. We also discuss **weakly summable Fredholm modules and the Dixmier trace** in the cases of some finitely and infinitely ramified fractals (including non-self-similar fractals) if the so-called spectral dimension is less than 2. In the finitely ramified self-similar case we relate the p-summability question with estimates of the **Lyapunov exponents** for harmonic functions and the behavior of the **pressure function**.

L.G. Rogers, *Estimates for the resolvent kernel of the Laplacian on p.c.f. self-similar fractals and blowups*. Trans. Amer. Math. Soc. 364 (2012)
L.G. Rogers, R.S. Strichartz, *Distribution theory on P.C.F. fractals.* , J. Anal. Math. **112** (2010)
M. Ionescu, L.G. Rogers, R.S. Strichartz *Pseudo-differential Operators on Fractals* arXiv:1108.2246

J. Kigami, *Resistance forms, quasisymmetric maps and heat kernel estimates*. Mem. Amer. Math. Soc. 216 (2012)
Volume doubling measures and heat kernel estimates on self-similar sets Mem. Amer. Math. Soc. 199 (2009)
Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate Math. Ann., vol 340 (2008)

N. Kajino, fine energy measure heat kernel and Hausdorff dimension estimates

S. Aaron, Z. Conn, R. Strichartz, H. Yu *Hodge-de Rham Theory on Fractal Graphs and Fractals* arXiv:1206.1310
M. Hinz *1-forms and polar decomposition on harmonic spaces*, Potential Analysis (2012)
Limit chains on the Sierpinski gasket. Indiana Univ. Math. J. 60 (2011)

F. Cipriani, D. Guido, T. Isola, J.-L. Sauvageot, *Differential* $\mathbf{1}$*-forms, their integrals and potential theory on the Sierpinski gasket*, preprint

Vector analysis on fractals and applications

Michael Hinz, Alexander Teplyaev, Vector analysis on fractals and applications.
arXiv:1207.6375 to appear in Contemporary Mathematics
We start with a local regular Dirichlet form and use the framework of **1**-forms and derivations introduced by Cipriani and Sauvageot to set up some elements of a related vector analysis in weak and non-local formulation. This allows to study various scalar and vector valued linear and non-linear partial differential equations on fractals that had not been accessible before. Subsequently a stronger (localized, pointwise or fiberwise) version of this vector analysis can be developed, which is related to previous work of Kusuoka, Kigami, Eberle, Strichartz, Hino, Ionescu, Rogers, Röckner, Hinz, T, also Cheeger, Heinonen, Tyson, Koskela et al.

related works

Hinz, Röckner, T, *Vector analysis for local Dirichlet forms and quasilinear PDE and SPDE on fractals*, arXiv:1202.0743

M. Hino
Energy measures and indices of Dirichlet forms, with applications to derivatives on some fractals. Proc. Lond. Math. Soc. (3) 100 (2010)
Geodesic distances and intrinsic distances on some fractal sets.
Measurable Riemannian structures associated with strong local Dirichlet forms.
Upper estimate of martingale dimension for self-similar fractals, to appear in Probab. Theory Related Fields. arXiv:1205.5617

Introduction

$$\operatorname{div}(a(\nabla u)) = f \tag{1}$$

$$\Delta u + b(\nabla u) = f \tag{2}$$

$$i\frac{\partial u}{\partial t} = (-i\nabla - A)^2 u + Vu. \tag{3}$$

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \end{cases} \tag{4}$$

Navier-Stokes equations

Theorem (a Hodge theorem, Hinz-T)

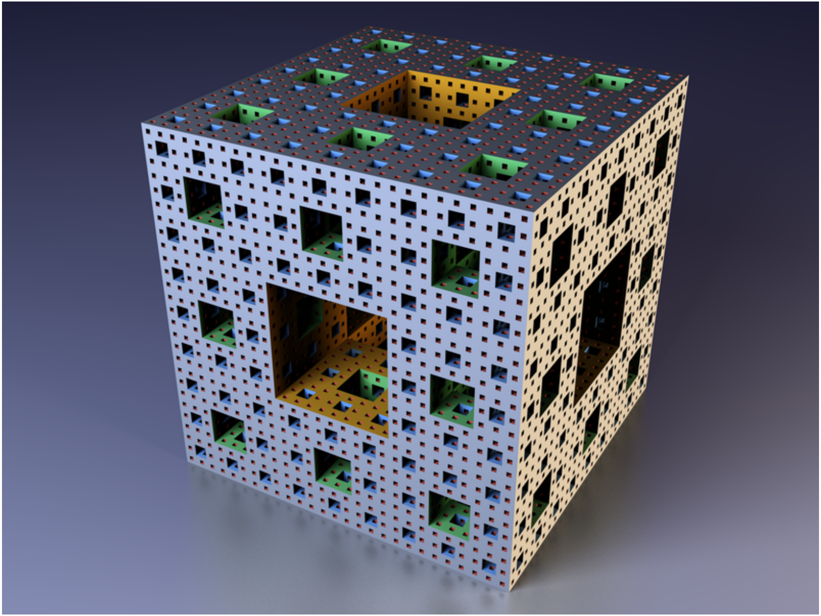
Assume that the space $\mathbf{X}$ *is compact, connected and topologically one-dimensional of arbitrarily large Hausdorff and spectral dimensions. A* $\mathbf{1}$*-form* $\omega \in \mathcal{H}$ *is harmonic if and only if it is in* $(\mathbf{Im}\,\partial)^{\perp}$*, that is* $\mathbf{div}\,\omega = \mathbf{0}$*.*

Using the classical identity $\frac{1}{2}\nabla|\mathbf{u}|^2 = (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times\mathbf{curl}\,\mathbf{u}$ we obtain

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \frac{1}{2}\partial\Gamma_{\mathcal{H}}(\mathbf{u}) - \Delta_1\mathbf{u} + \partial\mathbf{p} = \mathbf{0} \\ \partial^*\mathbf{u} = \mathbf{0}. \end{cases} \tag{5}$$

Theorem

Any weak solution $\mathbf{u}$ *of (5) is unique, harmonic and stationary (i.e.* $\mathbf{u_t} = \mathbf{u_0}$ *is independent of* $\mathbf{t} \in [\mathbf{0}, \infty)$*) for any divergence free initial condition* $\mathbf{u_0}$*.*



Theorem

Assume that points have positive capacity (i.e. we have a resistance form in the sense of Kigami) and the topological dimension is one. Then a nontrivial solution to (5) exists if and only if the first Čech cohomology $\check{\mathbf{H}}^{\mathbf{1}}(\mathbf{X})$ *of* $\mathbf{X}$ *is nontrivial.*

Remark

We conjecture that any set that carries a regular resistance form is a topologically one-dimensional space when equipped with the associated resistance metric.

$$\mathcal{E}^{a,V}(f,g) = \langle(-i\partial - a)f, (-i\partial - a)g\rangle_{\mathcal{H}} + \langle fV, g\rangle_{L_2(X,m)}, \quad f,g \in \mathcal{C}_{\mathbb{C}},$$

Theorem

Let $\mathbf{a} \in \mathcal{H}_\infty$ *and* $\mathbf{V} \in \mathbf{L}_\infty(\mathbf{X},\mathbf{m})$*.*

(i) *The quadratic form* $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ *is closed.*

(ii) *The self-adjoint non-negative definite operator on* $\mathbf{L}_{2,\mathbb{C}}(\mathbf{X},\mathbf{m})$ *uniquely associated with* $(\mathcal{E}^{a,V}, \mathcal{F}_{\mathbb{C}})$ *is given by*

$$\mathbf{H}^{a,V} = (-i\partial - a)^*(-i\partial - a) + \mathbf{V},$$

and the domain of the operator $\mathbf{A}$ *is a domain of essential self-adjointness for* $\mathbf{H}^{a,V}$*.*

Note: related Dirac operator is well defined and self-adjoint

$$\mathbf{D} = \begin{pmatrix} 0 & -i\partial^* \\ -i\partial & 0 \end{pmatrix}$$

Dirichlet forms and energy measures

Let $\mathbf{X}$ be a locally compact separable metric space and $\mathbf{m}$ a Radon measure on $\mathbf{X}$ such that each nonempty open set is charged positively. We assume that $(\mathcal{E}, \mathcal{F})$ is a symmetric local regular Dirichlet form on $\mathbf{L_2(X, m)}$ with core $\mathcal{C} := \mathcal{F} \cap \mathbf{C_0(X)}$. Endowed with the norm $\|\mathbf{f}\|_{\mathcal{C}} := \mathcal{E}(\mathbf{f})^{1/2} + \sup_{\mathbf{X}} |\mathbf{f}|$ the space $\mathcal{C}$ becomes an algebra and in particular,

$$\mathcal{E}(\mathbf{fg})^{1/2} \leq \|\mathbf{f}\|_{\mathcal{C}} \|\mathbf{g}\|_{\mathcal{C}}, \quad \mathbf{f}, \mathbf{g} \in \mathcal{C}, \tag{6}$$

see [?]. For any $\mathbf{g}, \mathbf{h} \in \mathcal{C}$ we can define a finite signed Radon measure $\mathbf{\Gamma(g, h)}$ on $\mathbf{X}$ such that

$$2 \int_{\mathbf{X}} \mathbf{f} \, d\mathbf{\Gamma(g, h)} = \mathcal{E}(\mathbf{fg}, \mathbf{h}) + \mathcal{E}(\mathbf{fh}, \mathbf{g}) - \mathcal{E}(\mathbf{gh}, \mathbf{f}), \quad \mathbf{f} \in \mathcal{C},$$

the *mutual energy measure* of $\mathbf{g}$ and $\mathbf{h}$. By approximation we can also define the mutual energy measure $\mathbf{\Gamma(g, h)}$ for general $\mathbf{g}, \mathbf{h} \in \mathcal{F}$. Note that $\mathbf{\Gamma}$ is symmetric and bilinear, and $\mathbf{\Gamma(g)} \geq \mathbf{0}$, $\mathbf{g} \in \mathcal{F}$. For details we refer the reader to [?]. We provide some examples.

Examples

(i) *Dirichlet forms on Euclidean domains*. Let $\mathbf{X} = \mathbf{\Omega}$ be a bounded domain in $\mathbb{R}^{\mathbf{n}}$ with smooth boundary $\partial\mathbf{\Omega}$ and

$$\mathcal{E}(\mathbf{f}, \mathbf{g}) = \int_{\mathbf{\Omega}} \nabla \mathbf{f} \nabla \mathbf{g} \, \mathbf{dx}, \quad \mathbf{f}, \mathbf{g} \in \mathbf{C}^{\infty}(\mathbf{\Omega}).$$

If $\mathbf{H}_0^1(\mathbf{\Omega})$ denotes the closure of $\mathbf{C}^{\infty}(\mathbf{\Omega})$ with respect to the scalar product $\mathcal{E}_1(\mathbf{f}, \mathbf{g}) := \mathcal{E}(\mathbf{f}, \mathbf{g}) + \langle \mathbf{f}, \mathbf{g} \rangle_{\mathbf{L}_2(\mathbf{\Omega})}$, then $(\mathcal{E}, \mathbf{H}_0^1(\mathbf{\Omega}))$ is a local regular Dirichlet form on $\mathbf{L}_2(\mathbf{\Omega})$. The mutual energy measure of $\mathbf{f}, \mathbf{g} \in \mathbf{H}_0^1(\mathbf{\Omega})$ is given by $\nabla \mathbf{f} \nabla \mathbf{g} \mathbf{dx}$.

(ii) *Dirichlet forms on Riemannian manifolds*. Let $\mathbf{X} = \mathbf{M}$ be a smooth compact Riemannian manifold and

$$\mathcal{E}(\mathbf{f}, \mathbf{g}) = \int_{\mathbf{M}} \langle \mathbf{df}, \mathbf{dg} \rangle_{\mathbf{T}^*\mathbf{M}} \, \mathbf{dvol}, \quad \mathbf{f}, \mathbf{g} \in \mathbf{C}^{\infty}(\mathbf{M}).$$

Here **dvol** denotes the Riemannian volume measure. Similarly as in (i) the closure of $\mathcal{E}$ in $\mathbf{L}_2(\mathbf{M}, \mathbf{dvol})$ yields a local regular Dirichlet form. The mutual energy measure of two energy finite functions $\mathbf{f}, \mathbf{g}$ is given by $\langle \mathbf{df}, \mathbf{dg} \rangle_{\mathbf{T}^*\mathbf{M}} \, \mathbf{dvol}$.

(iii) *Dirichlet forms induced by resistance forms on fractals*.

1-forms and vector fields

Consider $\mathcal{C} \otimes \mathcal{B}_b(X)$, where $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X with the symmetric bilinear form

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := \int_X bd \, d\Gamma(a, c), \tag{7}$$

$a \otimes b, c \otimes d \in \mathcal{C} \otimes \mathcal{B}_b(X)$, let $\|\cdot\|_{\mathcal{H}}$ denote the associated seminorm on $\mathcal{C} \otimes \mathcal{B}_b(X)$ and write
Define *space of differential 1-forms on* X

$$\mathcal{H} = \mathcal{C} \otimes \mathcal{B}_b(X) / \ker \ \|\cdot\|_{\mathcal{H}}$$

we
The space $\mathcal{H}$ becomes a bimodule if we declare the algebras $\mathcal{C}$ and $\mathcal{B}_b(X)$ to act on it as follows: For $a \otimes b \in \mathcal{C} \otimes \mathcal{B}_b(X)$, $c \in \mathcal{C}$ and $d \in \mathcal{B}_b(X)$ set

$$c(a \otimes b) := (ca) \otimes b - c \otimes (ab) \tag{8}$$

and

$$(a \otimes b)d := a \otimes (bd). \tag{9}$$

A *derivation operator* $\partial : \mathcal{C} \to \mathcal{H}$ can be defined by setting

$$\partial \mathbf{f} := \mathbf{f} \otimes \mathbf{1}.$$

It obeys the Leibniz rule,

$$\partial(\mathbf{f}\mathbf{g}) = \mathbf{f}\partial\mathbf{g} + \mathbf{g}\partial\mathbf{f}, \quad \mathbf{f}, \mathbf{g} \in \mathcal{C}, \tag{10}$$

and is a bounded linear operator satisfying

$$\|\partial \mathbf{f}\|_{\mathcal{H}}^2 = \mathcal{E}(\mathbf{f}), \quad \mathbf{f} \in \mathcal{C}. \tag{11}$$

On Euclidean domains and on smooth manifolds the operator ∂ coincides with the classical exterior derivative (in the sense of $\mathbf{L_2}$-differential forms). Details can be found in [**?**, **?**, **?**, **?**, **?**].

Being Hilbert, $\mathcal{H}$ is self-dual. We therefore regard $\mathbf{1}$-forms also as *vector fields* and ∂ as the *gradient operator*. Let $\mathcal{C}^*$ denote the dual space of $\mathcal{C}$, normed by

$$\|\mathbf{w}\|_{\mathcal{C}^*} = \sup\{|\mathbf{w}(\mathbf{f})| : \mathbf{f} \in \mathcal{C}, \|\mathbf{f}\|_{\mathcal{C}} \leq 1\}.$$

Given $\mathbf{f}, \mathbf{g} \in \mathcal{C}$, consider the functional

$$\mathbf{u} \mapsto \partial^*(\mathbf{g}\partial\mathbf{f})(\mathbf{u}) := -\langle \partial\mathbf{u}, \mathbf{g}\partial\mathbf{f}\rangle_{\mathcal{H}} = -\int_{\mathbf{X}} \mathbf{g}\, d\Gamma(\mathbf{u}, \mathbf{f})$$

on $\mathcal{C}$. It defines an element $\partial^*(\mathbf{g}\partial\mathbf{f})$ of $\mathcal{C}^*$, to which we refer as the *divergence of the vector field* $\mathbf{g}\partial\mathbf{f}$.

Lemma

The divergence operator ∂^* extends continuously to a bounded linear operator from $\mathcal{H}$ into $\mathcal{C}^*$ with $\|\partial^*\mathbf{v}\|_{\mathcal{C}^*} \leq \|\mathbf{v}\|_{\mathcal{H}}$, $\mathbf{v} \in \mathcal{H}$. We have

$$\partial^*\mathbf{v}(\mathbf{u}) = -\langle \partial\mathbf{u}, \mathbf{v}\rangle_{\mathcal{H}}$$

for any $\mathbf{u} \in \mathcal{C}$ and any $\mathbf{v} \in \mathcal{H}$.

The Euclidean identity

$$\mathbf{div}\,(\mathbf{g}\,\mathbf{grad}\,\mathbf{f}) = \mathbf{g}\boldsymbol{\Delta}\mathbf{f} + \nabla\mathbf{f}\nabla\mathbf{g}$$

has a counterpart in terms of ∂ and ∂^*. Let $(\mathbf{A}, \mathbf{dom}\ \mathbf{A})$ denote the infinitesimal $\mathbf{L}_2(\mathbf{X}, \mu)$-generator of $(\mathcal{E}, \mathcal{F})$.

Lemma

We have

$$\partial^*(\mathbf{g}\partial\mathbf{f}) = \mathbf{g}\mathbf{A}\mathbf{f} + \boldsymbol{\Gamma}(\mathbf{f}, \mathbf{g})\ ,$$

for any simple vector field $\mathbf{g}\partial\mathbf{f}$, $\mathbf{f}, \mathbf{g} \in \mathcal{C}$, *and in particular,* $\mathbf{A}\mathbf{f} = \partial^*\partial\mathbf{f}$ *for* $\mathbf{f} \in \mathcal{C}$.

Corollary

The domain $\mathbf{dom}\ \partial^*$ *agrees with the subspace*

$$\{\mathbf{v} \in \mathcal{H} : \mathbf{v} = \partial\mathbf{f} + \mathbf{w} : \mathbf{f} \in \mathbf{dom}\ \mathbf{A}\ ,\ \ \mathbf{w} \in \mathbf{ker}\ \partial^*\}\ .$$

For any $\mathbf{v} = \partial\mathbf{f} + \mathbf{w}$ *with* $\mathbf{f} \in \mathbf{dom}\ \mathbf{A}$ *and* $\mathbf{w} \in \mathbf{ker}\ \partial^*$ *we have* $\partial^*\mathbf{v} = \mathbf{A}\mathbf{f}$.