Diffusive limits on the Penrose tiling

A. Telcs
AFRT December 2012 Hong Kong

Supporting research schools at the University of Pannonia

Reg.No.:TÁMOP-4.2.2/B-10/1-2010-0025

Project period: 2011.08.01-2013.07.31.

Name and address of Contractor:

University of Pannonia

Egyetem utca 10, Veszprém H-8200

Name and Address of Subcontractor:

Agricultural Research Institute of

the Hungarian Academy of Sciences

Brunszvik utca 2, Martonvásár H-2462

Total funding granted by the European Union and the Hungarian Government:
669 779 225.- HUF.

The project is supported by the European Union.

HUNGARY'S RENEWAL

NEW SZÉCHENYI PLAN

Diffusive limits on the Penrose tiling

András Telcs

Hong Kong
December 2012.

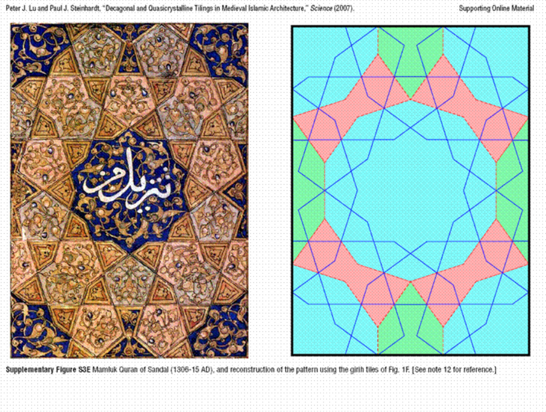
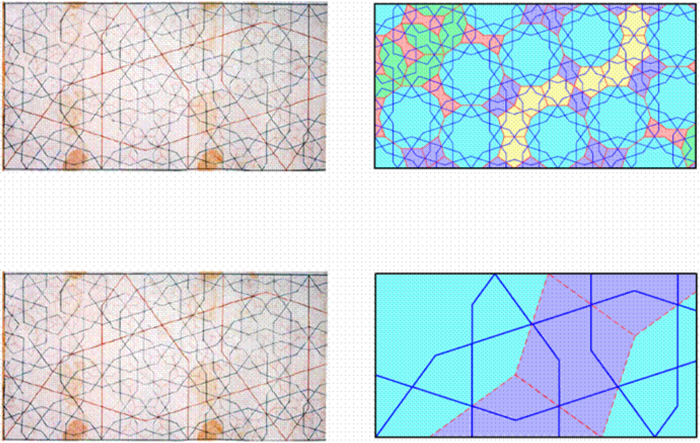


Figure: Islamic tiling from the 15th century

Peter J. Lu and Paul J. Steinhardt "Decagonal and Quasi-crystalline Tilings in Medieval Islamic Architecture," Science 315, 1106 (2007).

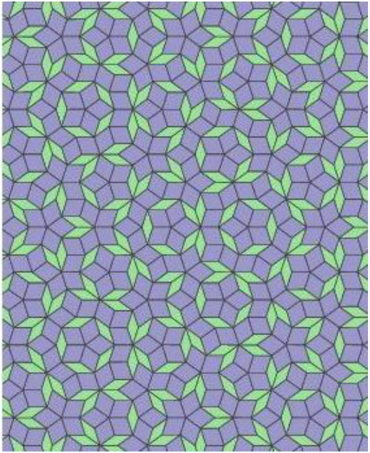
The beginings

Peter J. Lu and Paul J. Steinhardt, "Decagonal and Quasicrystalline Tilings in Medieval Islamic Architecture," *Science* (2007). Supporting Online Material



Supplementary Figure S4A Complete panel 28 from the Topkapi scroll (partly shown in Fig. 1G), where the red ink dots mark the boundaries of the girih tiles. In the upper reconstruction, the girih tiles have been filled in with color according to Fig. 1F. The thick red lines in the original scroll correspond to the strapwork decoration of girih tiles at a much larger length scale, shown in the lower reconstruction. This is a documented example of girih-tile subdivision, as each large girih tile transforms to the same corresponding pattern of small girih tiles. [See main text and note 13.]

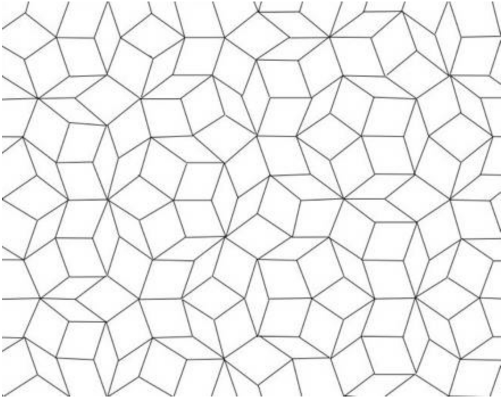
The Penrose tiling



Penrose 1978

Random walk on the Penrose graph

The graph and the net (the dual of the Penrose tiling)



The graph and the net (the dual of the Penrose tiling)

Γ countably infinite graph, with edges $x \sim y$, and trivial edge weight 1, degree of nodes, $d(x) = \sum_y I(x \sim y) = \frac{1}{4}$

$$\begin{aligned} P(x,y) &= \frac{1}{d(x)} \text{ if } x \sim y \\ \mathbb{P}(X_n = y | X_{n-1} = x) &= P(x,y) \end{aligned}$$

defines a $d(x)$-reverzibe Markov chain

Conventions

- Γ with $d(x,y)$, graph distance called Penrose graph
- Γ with $|x-y|$ embedded into $\mathbb{R}^2$ called Penrose net
- $\mathbb{Z}^2$ with $d(x,y)$ the graph on the integer lattice
- $\mathbb{Z}^2$ with $|x-y|$ embedded into $\mathbb{R}^2$ called integer net

Questions

Random walks on graphs

$$\Gamma \ \mu_{x,y} = 1, \ d(x,y) \ , \ p_n(x,y) \ ?$$

$$\frac{1}{Cn} e^{-C \frac{d^2(x,y)}{n}} \leq p_n(x,y) + p_{n+1}(x,y) \leq \frac{C}{n} e^{-\frac{d^2(x,y)}{Cn}}$$

Questions

Domokos Szász problem: Do we have invariance principle on the Penrose net?

$\Gamma \ \mu_{x,y} = 1, \ |x - y|$

$$\frac{1}{\sqrt{a}} X_{\lfloor at \rfloor} \Longrightarrow W_D(t) \ \text{ as } a \to \infty$$

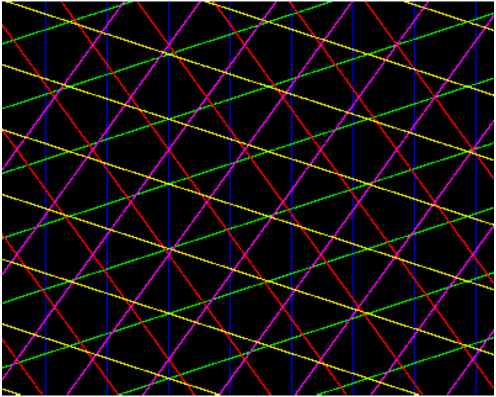
as $a \to \infty$. D pos. def.

Why is this a question?

- put $\mathbb{Z}^d$ grid on $\mathbb{R}^d$ then the invariance principle holds for the ssnn RW
- Put any periodic (shift invariant) grid on $\mathbb{R}^d$, still we have the IP.
- What if we put a non-periodic net on it?
- What if .... (comes later).

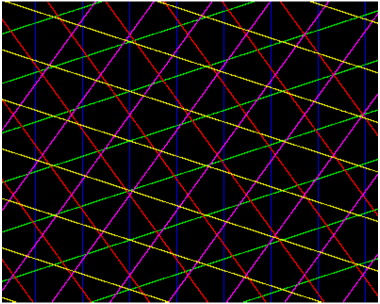
The construction of the Penrose tiling (deBruijn)

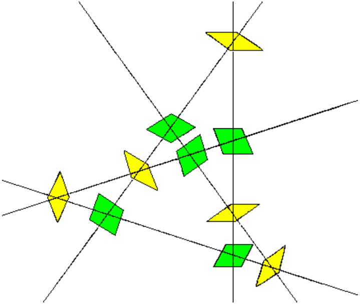
Pentagrid



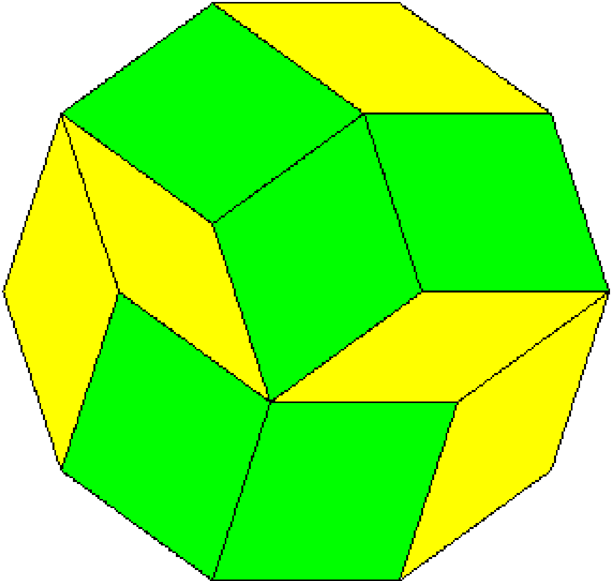
The construction of the Penrose tiling (deBruijn)

Pentagrid





Pentagrid



Rough isometry

Given Γ, Γ', rough isometric if , there are
$\Psi : \Gamma \to \Gamma', a, b, c, M > 0$:

$$\frac{1}{a} d(x, y) - b \le d'(\Psi(x), \Psi(y)) \le a d(x, y) + b,$$

$$\frac{1}{c} d'(\Psi(x)) \le d(x) \le c d'(\Psi(x)),$$

for all $y' \in \Gamma'$:

$$d'(\Psi(\Gamma), y') < M.$$

Let us use Ψ^{-1}.

Stability against rough isometry

Stability of $(GE_{\alpha,2})$

$$c\frac{1}{n^{\alpha/2}}\exp\left[-C\frac{d^2(x,y)}{t}\right]\leq\widetilde{p}_n(x,y)\leq C\frac{1}{n^{\alpha/2}}\exp\left[-c\frac{d^2(x,y)}{t}\right]$$

$$\widetilde{p}_n=p_n+p_{n-1},$$

Theorem

(Delmotte 1997.) Yes.

On $\mathbb{Z}^d$ we have,$(GE_{d,2})$

Random walk on the Penrose graph

Γ Penrose graph
If Γ roughly isometric to $\mathbb{Z}^2$, then

$$c\frac{1}{n}\exp\left[-C\frac{d^2(x,y)}{t}\right] \leq \widetilde{p}_n(x,y) \leq C\frac{1}{n}\exp\left[-c\frac{d^2(x,y)}{t}\right]$$

on the Penrose graph.

Rough isometry

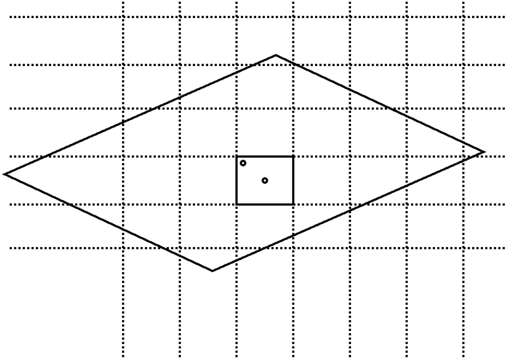
Theorem

Solomon The Penrose graph is bi-lipschitz to $\mathbb{Z}^2$.

Corollary

A Penrose graph is rough isometric to $\mathbb{Z}^2$*.*

Direct proof.Ψ :



Random walk on the Penrose graph

Theorem

$$c\frac{1}{n}\exp\left[-C\frac{d^2(x,y)}{t}\right] \leq \widetilde{p}_n(x,y) \leq C\frac{1}{n}\exp\left[-c\frac{d^2(x,y)}{t}\right]$$

Preparation

$$c\frac{1}{n}\exp\left[-C\frac{d^2(x,y)}{t}\right] \leq \widetilde{p}_n(x,y) \leq C\frac{1}{n}\exp\left[-c\frac{d^2(x,y)}{t}\right]$$

On the Penrose net

the random walk $X_n = \sum_{i=1}^{n} x_i$:x_i vectors between centers of tiles.
ω_n the environment seen from X_n
$Z_n = (\omega_n, X_n)$,

$$X_n = \sum_{i=1}^{n} V(Z_{i-1}, Z_i)$$

$$V(Z_{i-1}, Z_i) = X_i - X_{i-1} = x_i .$$

Penrose graph

Theorem

(deMasi, Ferrari, Goldstein, Wick) Z_n reversible ergodic Markov chain with μ stationary probability measure, X_n as above, anti-symmetric function of Z then

$$\frac{1}{\sqrt{A}} X_{\lfloor At \rfloor} \Longrightarrow W_D\left(t\right).$$

D ???.

The Penrose graph

Theorem

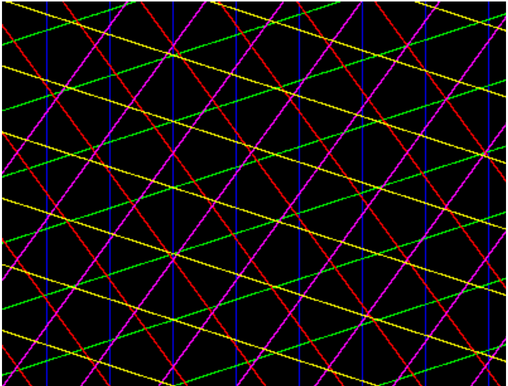
For the random walk on the Penrose net the invariance principle holds.

Proof.

1. Z_n egrodic μ stac. (M. Kunz,A. Robinson)
2. D pos. def. so the limiting process is non-degenerate.

Proof 1.

The five directions $e_k =$
$\left(\cos\left(\frac{\pi}{2} + (k-1)\frac{2\pi}{5}\right), \sin\left(\frac{\pi}{2} + (k-1)\frac{2\pi}{5}\right)\left(\frac{\pi}{2} + (k-1)\frac{2\pi}{5}\right)\right)$
$k = 1..5$ and the grids
$G_k = \{x \in \mathbb{R}^2 : xe_{k\perp} = z + \gamma_k, z \in \mathbb{Z}\}$



A cross-points $G_i \cap G_j$ are center of a tiles, the position is defned by the phases $\gamma_l, \gamma_m \mod 1$ so that

$$\sum_{i=1}^{5} \gamma_i = 0$$

and z.

Proof 1.

X_n is at a crosspoint of the pentagrid $G_i \cap G_j$: $i \neq j \in \{1, 2, 3, 4, 5\}$ that is the reference point for ω, i.e.: $\gamma_i = \gamma_j = 0$. For the pentagrid we know that

$$\sum_{i=1}^{5} \gamma_i = 0.$$

(γ_i mod 1) Still we have two "free" γ. So $\omega \leftrightarrow (i, j, \gamma_k, \gamma_l)$ identifies our position. We get 10 tori $\Omega_{i,j}$. Let $\Omega = \cup \Omega_{i,j}$. The dinamics: $\omega_n \rightarrow \omega_{n+1}$ on Ω.

Proof 1.

$\{\omega_n\} \subset \Omega$ is dense with null Lebesgue measure on the compact Ω. Let $\omega \in \Omega_{i,j}, |i-j| = 1, \omega' \in \Omega_{i,j}, |i-j| = 2$,

$$\begin{aligned} d\mu(\omega) &= \tau d\lambda(\omega) \\ d\mu(\omega') &= d\lambda(\omega') \end{aligned}$$

τ the golden ratio. μ is finite, can be normalised to probability measure. From the density theorem of topological groups it follows that if $A \subset \Omega$ invarian and $\mu(A) > 0$ then $\mu(A) = 1$,i.e. ergodic.

Proof 2.

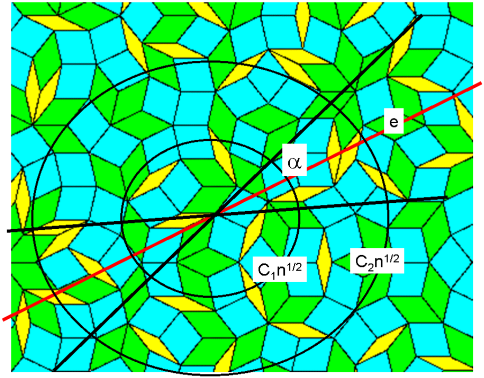
D non-degenerate. $0 \neq e \in \mathbb{R}^2$

$$e^* D e > 0.$$

Proof 2.

D non-degenerate. $0 \neq e \in \mathbb{R}^2$

$$e^* De > 0.$$



Proof 2.

D non-degenerate. $0 \neq e \in \mathbb{R}^2$

$$e^* De > 0.$$

Let $A = B\left(x_0, C_2\sqrt{n}\right) \setminus B\left(x_0, C_1\sqrt{n}\right)$, C cone at e with angle $\alpha : \pi/2 > \alpha > 0$. The intersection $H = A \cap C$.

$$\begin{aligned} & \mathbb{E}\left(e^* X_n X_n^* E e | X_0 = x_0\right) \\ = \; & \mathbb{E}\left(\left(e^* X_n\right)^2 | X_0 = x_0\right) \end{aligned}$$

Proof 2.

Let $A = B\left(x_0, C_2\sqrt{n}\right) \setminus B\left(x_0, C_1\sqrt{n}\right)$, C cone about x_0,with $\pi/2 > \alpha > 0$. $H = A \cap C$.

$$
\begin{aligned}
& \mathbb{E}\left(e^* X_n X_n^* e | X_0 = x_0\right) \\
= \ & \mathbb{E}\left(\left(e^* X_n\right)^2 | X_0 = x_0\right) \geq \sum_{x \in H} (ex)^2 P_n(x_0, x) \\
\geq \ & c\left(\cos(\alpha)\sqrt{n}\right)^2 \frac{c' \exp\left[-C \frac{\left(aC_2\sqrt{n}+b\right)^2}{n}\right]}{n} \geq c > 0
\end{aligned}
$$

$P_N \simeq P_G$!

Other tilings?

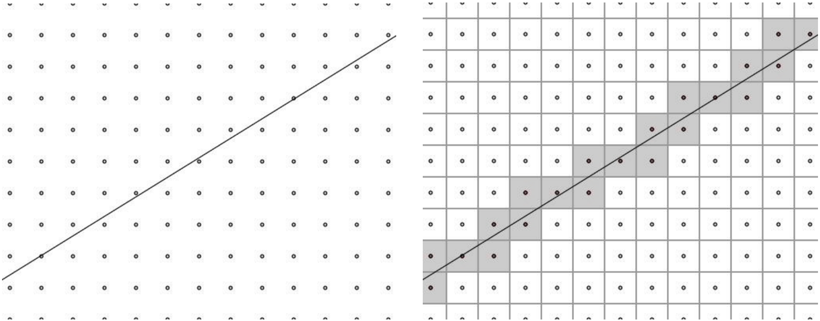
What conditions needed for the IP?

Quenched invariance principle

Is the invariance principle true for almos all starting point?

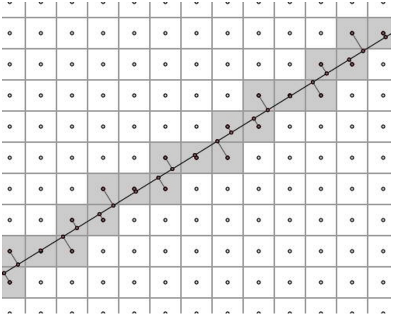
The construction of the Penrose tiling (deBruijn)

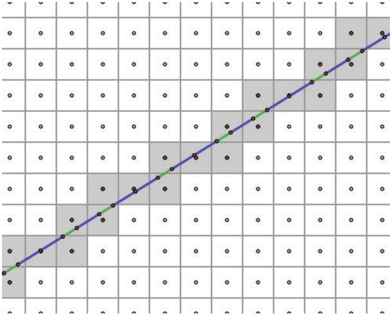
Pentagrid



The construction of the Penrose tiling (deBruijn)

Pentagrid





The construction of the Penrose tiling (deBruijn)

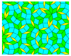
Pentagrid

$$\mathbb{Z}^5 \subset \mathbb{R}^5$$

Let $\zeta = \exp(2\pi i/5)$, the plain $\mathcal{P}$ consists of points $x, z \in \mathbb{R}^5$: orthogonal to $1 = (1,1,1,1,1)$ and $a_\pm = \left(1, \zeta^{\pm 2}, \zeta^{\pm 4}, \zeta^{\pm 6}, \zeta^{\pm 8}\right)$ $\mathcal{K}$ is the set of unit cubes in R^5 identified with their centre:

$$\mathcal{C} = \{K \in \mathcal{K} : K \cap \mathcal{P} \neq \emptyset\}$$

and project vertices and edges to $\mathcal{P}$.

Thanks for the attention! 

Local modification

Let Γ' a local modification of the Penrose graph,
Clearly Γ and Γ' rough isometric,$(GE_{2,2})$ holds:

$$\frac{c}{n}\exp\left(-C\frac{d^2\left(x,y\right)}{n}\right) \leq \overline{p}'_n\left(x,y\right) \leq \frac{C}{n}\exp\left(-c\frac{d^2\left(x,y\right)}{n}\right)$$

Local modification

Does the invariance hold? (Doma Szász's question)

Further questions

Is it true that $D = \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$?
For local modification is D invariant?
Is the invariance principle stabil agains rough isometry?
How the percolation behaves for the Penrose graph?
And the RW on it??