Fractal tiles and quasidisks

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Are (the interiors) of disk-like fractal tiles quasidisks?

Fractal tiles

(a) self-affine tiles: $T = T(A, \mathcal{D})$ — the compact set satisfying

$$T = \bigcup_{d \in \mathcal{D}} A^{-1}(T + d)$$

with $A \in M(2, \mathbb{R})$ expanding, (|eigenvalues| > 1), digit set $\mathcal{D} = \{d_i, i = 0, \ldots, N-1\} \subset \mathbb{R}^2$, $|\det(A)| = N$ and $T^\circ \neq \emptyset$.

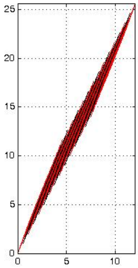


Figure: A disk like self-affine tile $T = T(A, D)$: $A = [0, 1; -15, 8]$, $\mathcal{D} = \{d_i = (i, 0)^t, i = 0, \ldots, 14\}$.

(b) Self-similar tiles:

$$T = \bigcup_{i=0}^{N-1} f_i(T) = \bigcup_{i=0}^{N-1} [r_i R_i(T) + b_i],$$

where the contraction ratios $r_i \in (0, 1)$, R_i orthogonal, $b_i \in \mathbb{R}^2$, $\{f_i\}$ satisfies the OSC, and $T^\circ \neq \emptyset$.

Quasidisk

(a) $S \subset \mathbb{R}^2$ — open bounded simply connected.
$[a, b]$ — **(rectilinear) cross-cut** of S.
V — the smaller half (smaller diameter) of $S \setminus [a, b]$.
If there is a $K > 0$ such that for all crosscut $[a, b]$ and V,

$$\frac{\operatorname{diam} V}{|a - b|} \leq K,$$

S is a **John Domain**.

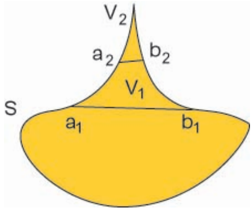


Figure: not a John domain.

(b) If there is a $K > 0$ such that for all $c, d \in S$,

$$\frac{\inf\{\mathrm{diam}(\widehat{cd}) : \widehat{cd} \subset S\}}{|c-d|} \le K,$$

then S is a **linearly connected domain**.

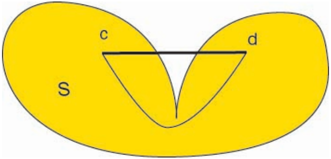


Figure: not a linearly connected domain.

(c) **quasidisk** — both John and linearly connected.

Quasidisks have many characterizing properties. e.g. Gehring (1982).

- Geometric properties: uniform domain, ∂T is a quasicircle, etc.

- Function theoretic properties: Sobolev extension domain, BMO extension domain.

Results

Theorem 1. A self-affine tile need not be a quasidisk.

Results

- T — a self-similar tile.
- $\mathcal{T}$ — a tiling constructed by blowing up T by an $f \in$ IFS. ($\mathcal{T} = \{f^{-k}(\text{level-}k \text{ pieces of } T), k = 1, 2, \ldots\}$.)
- vertex of $\mathcal{T}$ — a point in $\mathbb{R}^2$ belonging to ≥ 3 tiles in $\mathcal{T}$.

Theorem 2. Suppose $m := \inf\{\text{dist}(u, v), u, v \text{ vertices of } \mathcal{T}\} > 0$. Then T is a quasidisk.

Corollary $\mathcal{T}$ periodic or quasi-periodic $\Rightarrow$ T is a quasidisk.

Proof of Theorem 1: not all SA tiles are quasidisks

The higher level pieces can get sharper and sharper.

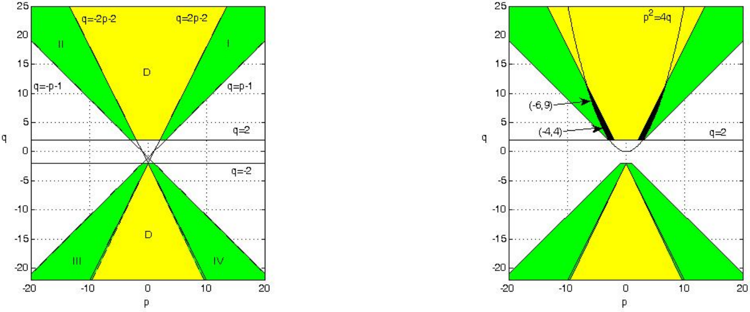
Hence not John.

Find an integral planar self-affine tile with consecutive collinear digit set that's not a quasidisk.

$p, q \in \mathbb{Z}$ such that
$A = [0, 1; -q, -p]$ expanding,
$\mathcal{D} = \{0, d_1, \ldots, d_{|q|-1}\}$, $d_i = (i, 0)^t$,

$$T = T(A, \mathcal{D}) = \bigcup_{i=0}^{|q|-1} A^{-1}(T + d_i).$$

T is disklike iff $|2p| \leq |q + 2|$. (Leung-Lau 2007)



(a) (b)

Figure: (a) Yellow: the (p, q)'s with disklike tiles, Green: non-disklike tiles. (b) Inside the parabolic region: A has complex eigenvalues.

Our example: $(p, q) = (-8, 15)$

Polygonal approx of disklike integral SA tiles (A having real eigenvalues.

Let

$$\begin{aligned} p_0 &= (0,0) \\ p_1 &= \frac{2q(q-1)}{(p^2+p\sqrt{p^2-4q}-2q)(p+q+1)}\left(1, \frac{-p-\sqrt{p^2-4q}}{2}\right) \\ p_2 &= (q-1)(A-I)^{-1}d_1 = \frac{q-1}{p+q+1}(-p-1, q) \\ p_3 &= p_2 - p_1 \end{aligned}$$

$T = T(A, \mathcal{D}) \subset$ closed bounding parallelogram P with vertices p_0, p_1, p_2, p_3.

Sides parallel to $A^{-1}d_1$ and 'the large eigendirection'.

$p_0, p_2 \in T$.

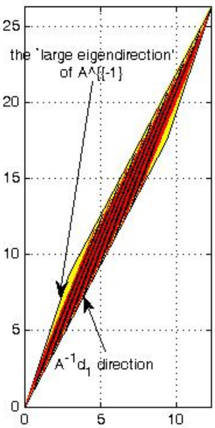


Figure: The bounding parallelogram P of $T = T(A, \mathcal{D})$, where $A = [0, 1; -15, 8]$, $\mathcal{D} = \{d_i = (i, 0)^t, i = 0, \ldots, 14\}$

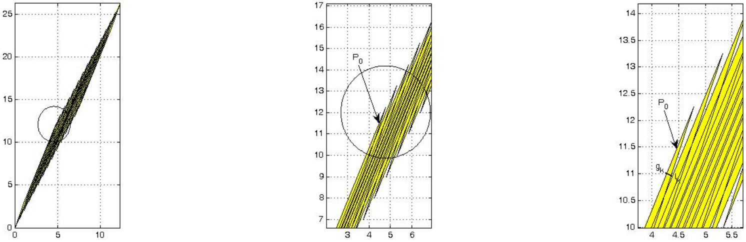
Iterate to get higher level polygonal approximations.

$$\begin{aligned}\mathcal{F}^k(P) &= \bigcup_{i_1,\ldots,i_k=0}^{14} A^{-k}P + i_k A^{-k} d_1 + \ldots + i_1 A^{-1} d_1 \\ &:= \bigcup_{i_1,\ldots,i_k=0}^{14} P_{i_1 \cdots i_k}.\end{aligned}$$

$P_{i_1 \ldots i_k}$ — level-k parallelograms;

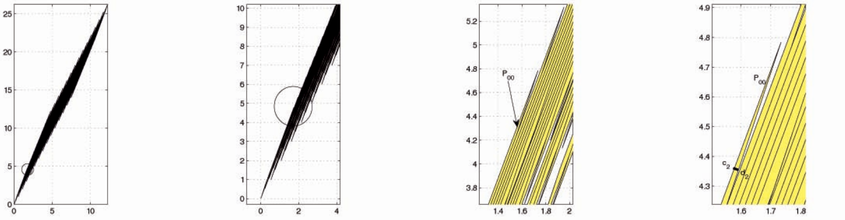
sides of $P_{i_1 \ldots i_k}$ — parallel to $v =$'the large eigendirection' of A^{-1}, and $A^{-k}d_1$(direction $\to v$) and

$\mathcal{F}^k(P)$— the level-k approx. of T; $\mathcal{F}^k(P) \subset \mathcal{F}^{k-1}(P)$.



(a) $\mathcal{F}^1(P)$ (b) zoom... (c) zoom further

Figure: (a) The level-1 approx $\mathcal{F}^1(P)$. (b) Zoom. The level-1 parallelogram $P_0 \subset \mathcal{F}^1(P)$ has its tip exposed.



(a) $\mathcal{F}^2(P)$ (b) Zoom (c) Zoom further (d) Tip of P_{00} exposed

Figure: The level-2 approximation $\mathcal{F}^2(P)$ of T. The level-2 parallelogram $P_{00} \subset \mathcal{F}^2(P)$ has its tip exposed.

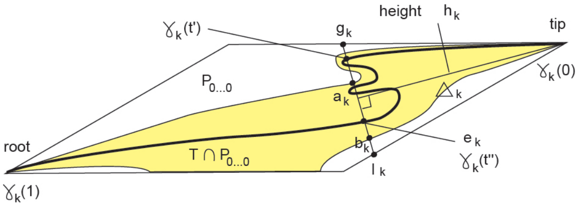


Figure: Inside the level-k parallelogram $P_{0\ldots0} \subset \mathcal{F}^k(P)$.
$\frac{\operatorname{diam} V^k}{|a_k - b_k|} \geq \frac{h_k}{|g_k - \ell_k|} \to \infty$.

(a) The sides of the level-k parallelogram $P_{0\ldots0} \subset \mathcal{F}^k(P)$ are parallel to $A^{-k}d_1$ and v_1, the 'large eigendirection' of A^{-1}.
(b) the direction of $A^{-k}d_1 \to$ the direction of v_1 as $k \to \infty$.

Proof of Theorem 2.

- T — self-similar tile.
- $\mathcal{T}$ — the (partial) tiling constructed by blowing up T by an $f \in$ IFS. ($\mathcal{T} = \{f^{-k}(\text{level-}k \text{ pieces of } T), k = 1, 2, \dots\}$.)

Theorem 2. Suppose $m := \inf\{\text{dist}(u, v), u, v \text{ vertices of } \mathcal{T}\} > 0$. Then T is a quasidisk.

Terminology, convention.

- For simplicity, assume constant contraction ratio r.
- $D := \operatorname{diam} T$.
- A **patch** $\mathcal{P}$ of $\mathcal{T}$:

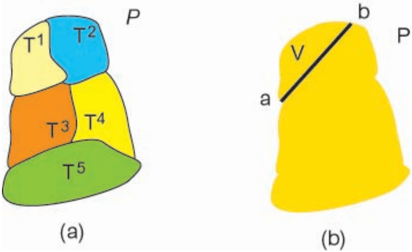


Figure: (a) A patch is a collection of tiles $\mathcal{P} \subset \mathcal{T}$, and (b) sometimes also refer to their union $P = \cup_{T \in \mathcal{P}} T$.

- **cross-cut of a disk-like patch**; **the smaller half** V of $P^\circ \setminus [a, b]$.

Hypothesis (H)(a property of $\mathcal{T}$ or equivalently T.)

There is a $\theta > 0$ such that for any disklike patch $\mathcal{P}$ and any cross-cut $[a, b]$ of P° with $|a - b| \leq \theta$,

- (H1) the smaller half V of $P^\circ \setminus [a, b]$ does not contain the entire interior of a tile, and
- (H2) the tiles $T' \in \mathcal{P}$ with $(T')^\circ \cap [a, b] \neq \emptyset$ share a common vertex.

‘Simplest’ appearances of Hypothesis (H):

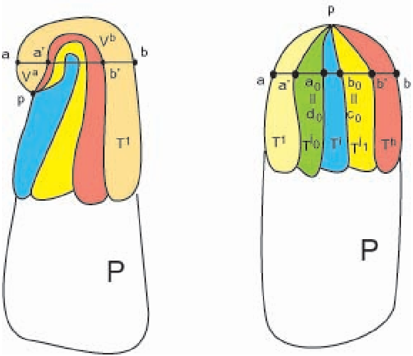


Figure: (H1) the resulting smaller half does not contain (the interior of) a whole tile, and (H2) tiles with interior intersecting the crosscut share a common vertex.

Consequence of Hypothesis (H):
a bound for $\mathrm{diam}(V)$: $|a-b| \leq \theta \Rightarrow \text{diam}(V) \leq 2D$.

(H1) $\Rightarrow V \subset \cup\{T' \in \mathcal{P} : (T')^\circ \cap [a,b] \neq \emptyset\}$. Then (H2) $\Rightarrow$ $\mathrm{diam} V \leq 2D$.

For really short cross-cuts $[a,b]$, blow-up the whole patch before using this estimate to get a really good bound on $\mathrm{diam}(V)$:

$$|a-b| \leq r^n\theta \Rightarrow \text{diam}(V) \leq 2r^n D.$$

A 2-step proof of Theorem 2

- Positive minimal vertex distance:
  $m := \inf\{\text{dist}(u, v), u, v \text{ vertices of } \mathcal{T}\} > 0$.

- Select θ so that
  (i) $\theta < m/3$;
  (ii) when a cross cut $[a, b]$ of a tile T' is of length $|a - b| \le \theta$, the smaller half V of $T' \setminus [a, b]$ has $\mathrm{diam}(V) < m/4$. (follows from disklikeness.)

Proposition 1 Positive minimal vertex distance $m > 0 \Rightarrow \mathcal{T}$ satisfies Hypothesis (H). In particular, (H1) and (H2) holds with the above choice of θ.

Proposition 2 $\mathcal{T}$ satisfies Hypothesis (H) $\Rightarrow$ T is a quasidisk.

Proof of Prop. 2: hypothesis (H)$\Rightarrow$ quasidisk

(i) Hypothesis (H) $\Rightarrow$ John domain:

$\mathcal{C}$ — the set of all cross-cuts of T.

Subclasses:

$$
\begin{aligned}
\mathcal{C}_0 &:= \{[a,b] \in \mathcal{C} : r\theta < |a-b|\}, \quad r \text{ — contraction ratio} \\
\mathcal{C}_1 &:= \{[a,b] \in \mathcal{C} : r^2\theta < |a-b| \le r\theta\} \\
&\;\;\vdots \\
\mathcal{C}_k &:= \{[a,b] \in \mathcal{C} : r^{k+1}\theta < |a-b| \le r^k\theta\}, \quad k \ge 1, \\
&\;\;\vdots
\end{aligned}
$$

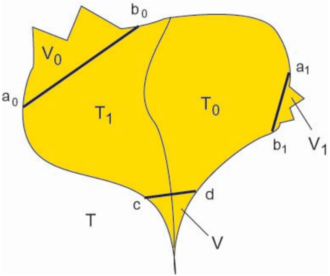


Figure: How Hypothesis (H) helps to control the ratio.

$[a_0, b_0] \in \mathcal{C}_0$, $\frac{\operatorname{diam} V_0}{|a_0 - b_0|} \leq \frac{D}{r\theta}$;
$[a_1, b_1] \in \mathcal{C}_1$, $\frac{\operatorname{diam} V_1}{|a_1 - b_1|} = \frac{\operatorname{diam} V_0}{|a_0 - b_0|} \leq \frac{D}{r\theta}$;
$[c, d] \in \mathcal{C}_1$, $\frac{\operatorname{diam} V}{|c - d|} = \frac{\operatorname{diam} f^{-1} V}{|f^{-1}[c,d]|} \leq \frac{2D}{r\theta}$, by the consequence of hypothesis (H).

$k \geq 1$:
$[a_k, b_k] \in \mathcal{C}_k$, entirely contained in some level-k piece of T: magnified k times (apply f^{-k}) to get

$$\frac{\mathrm{diam}(V)}{|a_k - b_k|} = \frac{\mathrm{diam} f^{-k}(V)}{|f^{-k}[a_k, b_k]|} \leq \frac{D}{r\theta};$$

$[c, d] \in \mathcal{C}_k$, intersecting the interior of ≥ 2 level-k pieces: magnify k times to get a cross-cut of length $\leq \theta$ of a disklike patch.

$$\frac{\mathrm{diam}(V)}{|c - d|} = \frac{\mathrm{diam} f^{-1}(V)}{|f^{-k}[c, d]|} \leq \frac{2D}{r\theta},$$

by the consequence of hypothesis (H).

Hence $\{ratios\}$ bounded, $\Rightarrow$ John.

Step (ii): Similar argument $\Rightarrow$ linearly connected.

Prop. 2 proved.

Proof of Prop 1: $m > 0 \Rightarrow$ hypothesis (H)

Recall:

- Positive minimal vertex distance:
  $m := \inf\{\text{dist}(u, v), u, v \text{ vertices of } \mathcal{T}\} > 0.$

- Select θ so that
  (i) $\theta < m/3$;
  (ii) when a cross cut $[a, b]$ of a tile T' is of length $|a - b| \leq \theta$, the smaller half V of $T' \setminus [a, b]$ has $\mathrm{diam}(V) < m/4$.
  (iii) $\mathrm{diam}(T') \geq m$ (as $\partial T'$ has ≥ 2 vertices).
  (iv) $\mathrm{diam}(T' \setminus \overline{V}) > 3m/4$

This θ guarantees (H2) vertex sharing. Example:

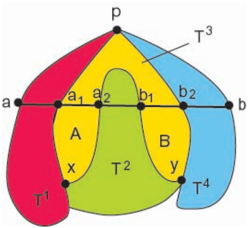


Figure: Suppose $|a - b| \leq \theta$. This picture is excluded by the choice of θ.

(a) A and B cannot be both the smaller halves of the cross-cuts $[a_1, a_2]$ and $[b_1, b_2]$ of T^3. (Otherwise, $|x - a_1|, |y - b_2| < m/4$, and $|a_1 - b_2| < |a - b| < m/3$, $\Rightarrow |x - y| < m$, contradiction.)

(b) Suppose $T^3 \setminus \overline{B}$ is the smaller half of $(T^3)^\circ \setminus [b_1, b_2]$. Then $p, x \in T^3 \setminus \overline{B} \Rightarrow |x - p| < m/4 < m$, contradiction.

How the choice of θ guarantees (H1): the smaller half of $P \setminus [a, b]$ does not contain an entire tile.

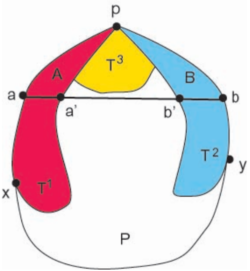


Figure: Suppose $|a - b| \leq \theta$. Then this picture is impossible.

(a) A, B are the smaller halves of $(T^1)^\circ \setminus [a_1, a_2]$ and $(T^2)^\circ \setminus [b_1, b_2]$. (Otherwise, a different pair of halves share a vertex.)
(b) The component C of $P^\circ \setminus [a, b]$ containing A and B has $\mathrm{diam}(C) = \mathrm{diam}(co(A, B)) \leq \mathrm{diam}(A) + \mathrm{diam}(B) < m/2$.
(c) $\mathrm{diam}(\mathrm{tile}) \geq m > 0$. Hence C can't contain an entire tile. (tile has ≥ 2 vertices on its boundary)

Thank you.