Intersection properties of random and deterministic measures

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Orthogonal projections

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Theorem (Marstrand, Mattila)

Let $A \subset \mathbb{R}^d$ be a Borel set, and let $s = \dim_H(A)$ be its Hausdorff dimension. If $s \leq k$ then the orthogonal projection onto almost all k-planes has dimension s, while if $s > k$, then the orthogonal projection of A onto almost all k-planes has positive k-dimensional Lebesgue measure.

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Motivation

For some (random) fractals, one would like to know more. In particular, if there are a.s. no exceptional directions for the projections.

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- Falconer and Grimmett (1992) showed that if the dimension of fractal percolation is > 1, the projections in the principal directions contain intervals a.s.
- This was vastly generalised by Rams and Simon (2011) who proved in a more general setting that all orthogonal projections onto lines have nonempty interior a.s. on dimension of the fractal percolation > 1.
- In case $s < 1$, Rams and Simon (2012) prove that the dimension is preserved under all orthogonal projections onto lines.
- Moreover, Peres and Rams have proved that in $\mathbb{R}^2$, all orthogonal projections of the fractal percolation measures in nonprincipal directions are absolutely continuous with a Hölder continuous density.

- For a closely related model in $\mathbb{R}^d$, Shmerkin and S. (2012) proved that if the dimension is $> k$, all orthogonal projections of the random limit measure onto k-planes are absolutley continuous with a uniformly bounded density. As a corollary, this settled a question of Carbery, Soria and Vargas on the dimension of sets which are not tube-null.

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- For fractal percolation, the (densities of the) projections in principal directions are easily seen to be a.s. discontinuous. Are there fractal measures of a given dimension $k < s < d$ such that all projections have a continuous density? If yes, how regular can the density be?

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- For which random fractals, can we prove the a.s. existence of scaled copies of arithmetic progressions and/or more general finite patterns?

- Projections are closely related to intersections.
- For instance, the orthogonal projection of $A \subset \mathbb{R}^d$ onto a plane $V \subset \mathbb{R}^d$ has nonempty interior, if and only if there is an open set $U \subset V$ such that the plane orthogonal to V through each point of U meets A.
- More generally, the continuity properties of the orthogonal projections of a measure μ are closely related to the fibers of μ along these planes (e.g. how fast does the total mass of the fiber change, when the fibre is moved).
- It turns out that this idea can be applied for intersections with many other families of sets and measures and not just for the intersections with affine planes and Hausdorff measures on them. For instance for the continuity of the intersections of certain random measures with respect to self-similar measures
- In many situations, the continuity results for the intersections of the random measures with a fixed deterministic family of measures can be used to deduce geometric information on the intersections of the random limit set with all sets in a given deterministic family.

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- There exists an increasing sequence of σ-algebras $\mathcal{B}_n$ such that μ_n is $\mathcal{B}_n$-measurable. Moreover, for all Borel sets B,

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Almost surely, the sequence μ_n is weakly convergent. Denote the limit by μ.

Intersections with deterministic measures

Let $\{\eta_t\}$, $t \in \Gamma$, be a family of measures indexed by a totally bounded metrix space (Γ, d) and let $\{\mu_n\}_n$ be a random martingale measure as in the previous slide. For all $t \in \Gamma$, and $n \in \mathbb{N}$, we define a measure μ_n^t as

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Sometimes, we also consider the measures μ^t defined as weak limits of μ_n^t.

Remarks

- For each fixed t it follows from the martingale condition that $|\mu^t|$ exists almost surely.
- For any two measures μ and ν, the method of slicing measures can be used to define the intersection of μ and almost all translates/isometric copies/homotethic copies etc. Our goal is to show that for certain random martingale measures and for many relevant families $\{\eta_t\}_{t\in\Gamma}$, the intersections are defined for all t and behave in a continuous way with respect to t.
- One might want to compare this with the classical results of Hawkes on the Hausdorff dimension of the intersections of a fixed Borel set A and almost all Brownian paths.

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- In this example, $d = 2$. Given some $k \in \mathbb{N}$, Γ is the family of all algebraic curves of degree at most k which intersect λ, d is the natural metric, and η_γ is length measure on $\gamma \cap \Omega$.
- Let $m \geq 2$, and let Γ be a totally bounded subset of uniformly contractive self similar IFSs with m maps. Suppose that each IFS $(g_1, \ldots, g_m) \in \Gamma$ satisfies the OSC. The measure $\eta_{(g_1,\ldots,g_m)}$ is then the natural self-similar measure for the corresponding IFS.

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A random martingale measure is called uniformly spatially independent if there is $C < \infty$ such that for all $n \in \mathbb{N}$, and for any $C2^{-n}$ separated family $\mathcal{U}$ of balls with radius 2^{-n}, the restrictions $\{\mu_{n+1}|_B \mid \mathcal{B}_n\}$ are independent.

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The following slightly weaker (but still very strong) notion of independence is often useful.

Definition

A random martingale measure is called spatially independent with respect to $\{\eta_t\}_{t\in\Gamma}$, if for all $t \in \Gamma$, some $C < \infty$ and all $n \in \mathbb{N}$, and for any $C2^{-n}$ separated family $\mathcal{U}$ of balls with radius 2^{-n}, the restrictions $\{\mu^t_{n+1}|_B \ |\mathcal{B}_n\}$ are independent.

Poissonian cut-out measures

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- For any Borel set $B \subset \mathbb{R}^d \times (0, 1)$, the random variable $|\mathcal{Y} \cap B|$ is Poisson with mean $\mathbf{Q}(B)$.
- If $\{B_j\}$ are pairwise disjoint subsets of $\mathbb{R}^d \times (0, 1)$, then the random variables $|\mathcal{Y} \cap B_j|$ are independent.

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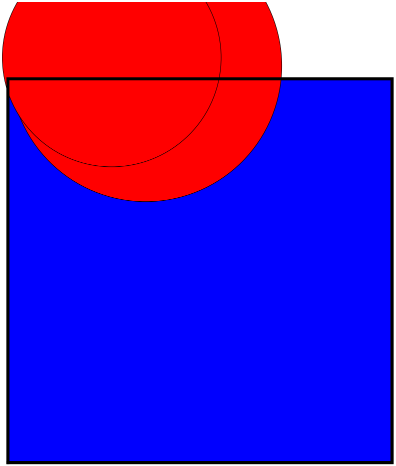
One can then form the random cut-out set

$$A = 2\Lambda_0 \setminus \bigcup_j \Lambda_{x_j, r_j},$$

where $\Lambda_{x,r}$ is the r-scaled copy of Λ_0 translated by x.

Ball type cut-out measures



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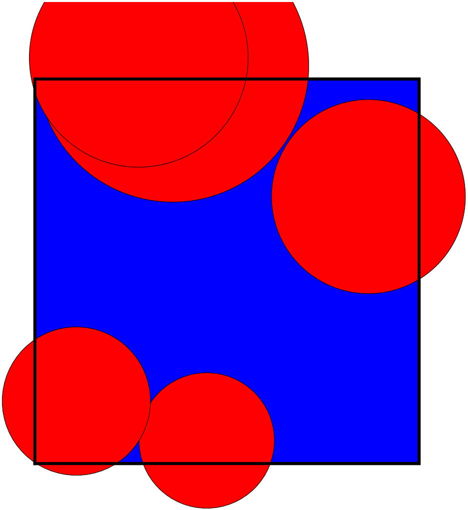

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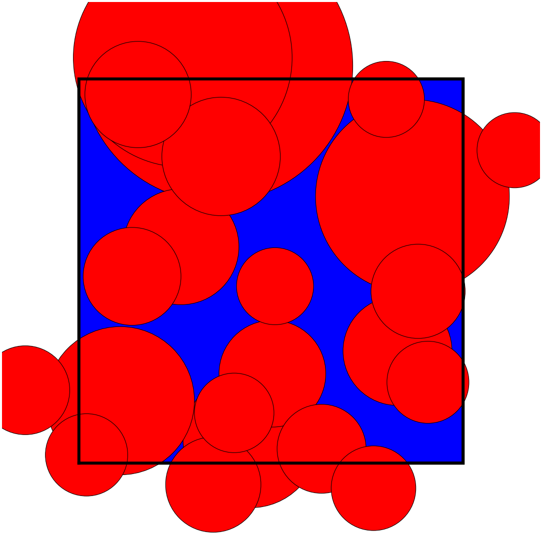

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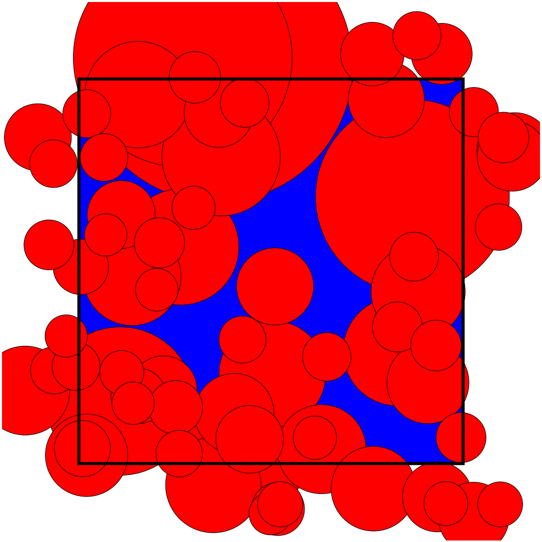

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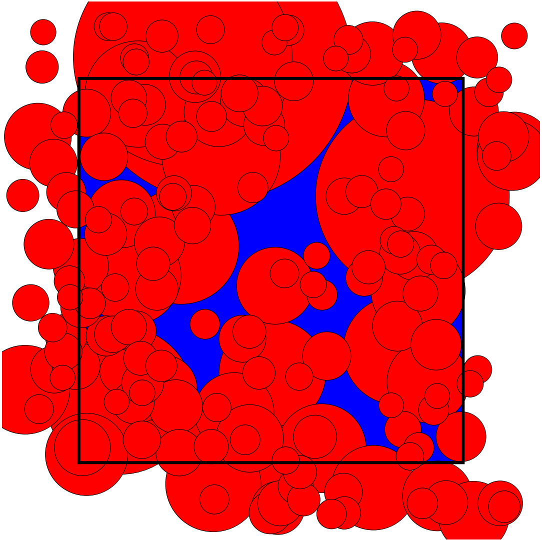

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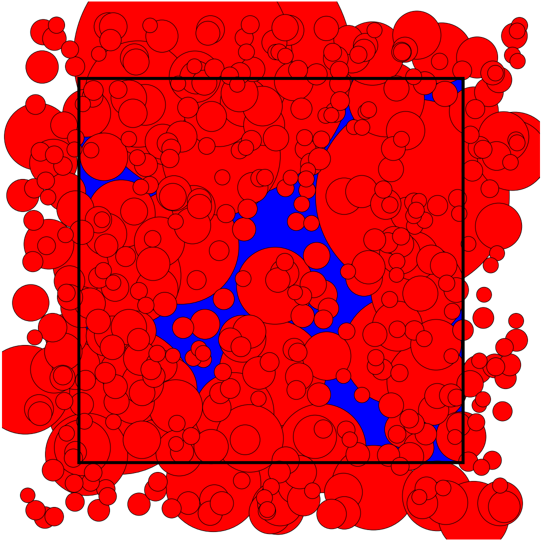

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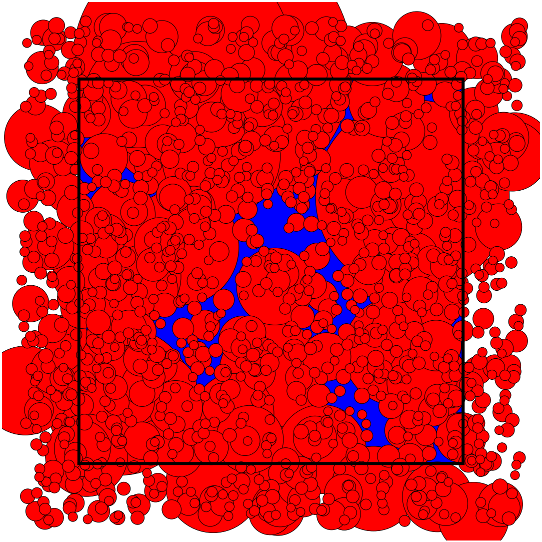

It turns out that there is a natural measure μ supported on A: it is the weak limit of the measures

$$\mu_n := 2^{\alpha n} \mathcal{L}|_{A_n} \,,$$

where $A_n = 2\Lambda_0 \setminus \bigcup \{\Lambda_{x_j, r_j} : r_j > 2^{-n}\}$, and $\alpha = \beta 2^{-d} c$, where c is the Lebesgue measure of Λ_0.

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More generally, let $\mathcal{X}$ denote the space of compact subsets of $\mathbb{R}^d$ and let $\mathbf{Q}$ be (an infinite) Borel measure on $\mathcal{X}$ satisfying:

- $\mathbf{Q}$ is translation invariant, i.e. $\mathbf{Q}(\mathcal{A}) = \mathbf{Q}(\{\Lambda + t : \Lambda \in \mathcal{A}\})$ for all $t \in \mathbb{R}$.
- $\mathbf{Q}$ is scale invariant, i.e. $\mathbf{Q}(\mathcal{A}) = \mathbf{Q}(\{s\Lambda : \Lambda \in \mathcal{A}\})$ for all $s > 0$, where $sA = \{sx : x \in A\}$.
- $\mathbf{Q}$ is locally finite, meaning that the $\mathbf{Q}$-measure of the family of all sets of diameter larger than 1 that are contained in $[-1,1]^d$ is finite.

Then one obtains a random martingale measure by considering a Poisson point process $\mathcal{Y} = \{\Lambda_j\}$ with intensity $\mathbf{Q}$ and setting

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where

$$A_n = \Omega \setminus \{\cup \Lambda_j \,:\, 2^{-n} \leq \operatorname{diam}(\Lambda_j) < 1\}\,,$$

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and $\mathbf{Q}_0$ is a measure supported on sets of diameter one such that $\mathbf{Q}$ is obtained as the push down of $\mathcal{L} \times \frac{dr}{tr} \times \mathbf{Q}_0$ under $(x, r, \Lambda) \to r(\Lambda + x)$.

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In many cases (e.g. for the ball-type or snowflake type cut-out measures) it can be verified that $\dim A$ (and $\dim \mu$) equals $d - \alpha$:

Theorem (See e.g. Thacker 2006, Nacu and Werner 2011)

Under mild geometric assumptions on the removed shapes, we have

$$\dim_B(A) = \dim_H(A) = d - \alpha \textit{ almost surely on } A \neq \emptyset$$

and moreover, $\dim(\mu, x) = d - \alpha$ *for* μ*-almost all* $x \in A$.

Subdivision random fractals

Let $F_0 \subset \mathbb{R}^d$ be a bounded closed set and let $\mathcal{F}_n$ be an increasing sequence of finite, atomic σ-algebras on F_0, with $\mathcal{F}_0 = \{F_0, \emptyset\}$. In the sequel we identify $\mathcal{F}_n$ with the collection of its atoms.For each $n \in \mathbb{N}$, let $c < p_n < 1$ and $0 \leq w(F) \leq C$, $F \in \mathcal{F}_n$ be random variables such that

- p_{n+1} and the $w(F)$, $F \in \mathcal{F}_{n+1}$, are $\mathcal{B}_n$-measurable, where $\mathcal{B}_n$ is the σ-algebra generated by $p_k, w(F)$ for $0 \leq k \leq n$ and $F \in \mathcal{F}_k$.
- $\mathbb{E}(w(F)|\mathcal{B}_n) = p_{n+1}$ for all $F \in \mathcal{F}_{n+1}$.

We define a sequence $\{\mu_n\}$ as follows:

- For each $F \in \mathcal{F}_n$, let $\mu_n|_F = \prod_{k=0}^n p_k^{-1} \prod_{k=0}^n w(F_k)$, where F_k is the atom of $\mathcal{F}_k$ which contains F.

Then $\{\mu_n\}$ is a random martingale measure. If, moreover:

- For any collection $\{F_j\}_j$ of atoms in $\mathcal{F}_n$ such that each $F \in \mathcal{F}_{n-1}$ contains at most one F_j, the random variables $w(F_j)$ are independent.

- There is $C > 0$ such that for all n and all $F \in \mathcal{F}_n$, there are at most C elements $F' \in \mathcal{F}_n$ such that $\operatorname{dist}(F, F') < 2^{-n}$,

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Example

Let $\mathcal{V}$ be a subcollection of the $(d-1)$-dimensional linear subspaces of $\mathbb{R}^d$ and suppose that the boundaries of each $F \in \mathcal{F}_n$ consist of at most C subsets that are parallel to elements of $\mathcal{V}$. Then we say that the Filtration is polygonal type.

Comparing Poissonian and subdivision models

- The main difference between the Poissonian cut-out model and the subdivision models are the scale and translation invariance properties. For the subdivision models, there can be a very limited scale and translation invariant (if any), arising from the nature of the filtrations $\mathcal{F}_n$ whereas the Poissonian cut-out model is translation and scale invariant inside the initial domain (often it is also rotationally invariant and sometimes even conformally invariant).
- On the other hand, the subdivision models have the advantage that there are no overlaps among the removed shapes of the same generation.
- Nevertheless, the Basic idea in both models is the same, they both give rise to a random martingale measures, and similar ideas can be used to study their geometric properties.

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- It depends on the particular problem, whether the necessary details are easier to carry on in the Poissonian cut-out or random subdivision fractal setting.

A general continuity result

Theorem (Shmerkin and S. 2012)

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a random martingale measure, and let $\{\eta_t\}_{t\in\Gamma}$ be a family of measures indexed by the metric space (Γ, d). We assume that there are constants $\theta, \gamma_0 > 0$ and $s > \alpha > 0$ such that:

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- *Almost surely, $\mu_n(x) \leq 2^{\alpha n}$ for all $n \in N$ and $x \in \mathbb{R}^d$.*

A general continuity result

Theorem (Shmerkin and S. 2012)

Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a random martingale measure, and let $\{\eta_t\}_{t\in\Gamma}$ be a family of measures indexed by the metric space (Γ, d). We assume that there are constants $\theta, \gamma_0 > 0$ and $s > \alpha > 0$ such that:

- *$\{\mu_n\}$ is spatially independent with respect to Γ.*
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- *Almost surely, there is a (random) integer N, such that*

$$\sup_{t,u\in\Gamma, t\neq u; n\geq N} (|\mu_n^t|_\infty - |\mu_n^u|_\infty) 2^{\theta n}\, d(t,u)^{\gamma_0} < \infty.$$

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Then there is $\gamma > 0$ (depending on $\theta, \gamma_0, \alpha, s$) such that almost surely $|\mu_n^t|_\infty$ converges uniformly in t, and the function $t \to |\mu^t|_\infty$ is Hölder continuous with exponent γ.

Hölder continuity of orthogonal projections

Applying the general continuity results for the ball-type cutout measures and letting Γ be the collection of all affine $(d-k)$-planes we are able to prove

Theorem (Shmerkin and S. 2012)

For all $k, d \in \mathbb{N}$ and $k < s \leq d$, there are measures μ in $\mathbb{R}^d$ such that

$$\dim(\mu, x) = s \textit{ for } \mu - \textit{almost all } x \in \mathbb{R}^d$$

and all orthogonal projections of μ onto k-planes are absolutely continuous with a $\gamma = \gamma(k, s)$ Hölder-continuous density

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In fact, the densities are shown to be jointly Hölder continuous with respect to (x, V), $x \in V$, $V \in \mathbf{G}_{d,k}$.

Intersections with algebraic curves

Applying the general continuity results for the snowflake-type cutout measures and letting $\mathcal{P}_k$ be the family of all real algebraic curves in the plane of degree at most k (with a natural metric), we arrive at the following result.

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Theorem (Shmerkin and S. 2012)

For each $1 < s < 2$, there are random martingale measures $\{\mu_n\}$ on $\mathbb{R}^2$ satisfying almost surely the following conditions

- $\dim(\mu, x) = s$ *for μ-almost all $x \in \mathbb{R}^2$,*
- *For all $k \in \mathbb{N}$ the sequence $2^{n(2-s)}\mathcal{H}^1(V \cap \mathrm{spt}(\mu_n))$ converges uniformly over all $V \in \mathcal{P}_k$, denote the limit by $|\mu^V|_\infty$.*
- $V \mapsto |\mu^V|_\infty$ *is Hölder continuous with exponent $\gamma = \gamma(s) > 0$.*
- $\sup_{V \in \mathcal{P}_k, n \in \mathbb{N}} 2^{n(2-s)}\mathcal{H}^1(V \cap \mathrm{spt}(\mu)(2^{-n})) < \infty$ *for all $k \in \mathbb{N}$. In particular, we have $\dim_B(V \cap \mathrm{spt}(\mu)) \leq s - 1$ for all algebraic curves V.*

Intersections with self-similar sets

Another application of the continuity result for the ball-type cutout measures and suitably chosen families $\{\eta_t\}_{t\in\Gamma}$ of self similar measures yields the following:

Theorem (Shmerkin and S. 2012)

For each $d \in \mathbb{N}$ and $0 < s < d$, there are random Borel sets $A \subset \mathbb{R}^d$ with $\dim_H(A) = \dim_B(A) = s$ such that for each self-similar set $E \subset \mathbb{R}^d$ satisfying the open set condition, we have

$$\dim_B(E \cap A) \leq \max\{0, \dim E + s - d\}.$$

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Remark

The continuity result behind the above Theorem is a generalisation of the first application, since each (compact subset of) affine k-plane is a subset of a self-similar set satisfying the open set condition.

Arithmetic sums of the random sets

A more general version of the continuity result (allowing dependencies) can be used to study the existence of finite patterns and the structure of the arithmetic sums. For instance:

Theorem (Shmerkin and S. 2012)

Let $d, m \in \mathbb{N}$ and let $A_1, \ldots, A_m \subset \mathbb{R}^d$ be m-independent ball type cut-out sets with scaling exponents $\alpha_1, \ldots, \alpha_d$ such that

$$\sum_{k=1}^{m} \alpha_k < d(m-1)\,.$$

Then, almost surely on each A_i being nonempty, the arithmetic sums

$$\sum_{k=1}^{m} \lambda_k A_k = \{\lambda_1 a_1 + \ldots + \lambda_m a_m \ : \ a_i \in A_i\}$$

have nonempty interior for all $0 \neq \lambda_1, \ldots, \lambda_m \in \mathbb{R}$.

Existence of finite patterns

Theorem (Shmerkin and S. 2012)

Fix $0 \neq a_1, \ldots, a_m \in \mathbb{R}^d$. *Let* $A \subset \mathbb{R}^d$ *be the ball-type random cut-out set with scaling exponent*

$$\alpha < \frac{d}{m}\,.$$

Then almost surely on $A \neq \emptyset$, *the set* A *contains the configuration*

$$y + \lambda a_1, y + \lambda a_2, \ldots, y + \lambda a_m \in A$$

for an open set of parameters $\lambda > 0$ *(with some* $y = y(\lambda) \in \mathbb{R}^d$*).*