New Development of Fractal PDE

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Abstract

Fractal PDE, as a quite new topic in the area of "Fractal Analysis", is developing very fast since the end of last century. It is motivated from physics, astronomy, geology, …, for instance, scientists hope mathematicians show the speed of an ant when it moves along a Weierstrass curve, or speed of Brownian motion. Moreover, what are solutions of a drum with a fractal boundary, and so on. Thus, the fractal PDE problems are proposed.

In this paper, we will show 4 important ideas to study the fractal PDE with corresponding main methods and main results. Finally, Some open problems are also indicated.

1. The idea is proposed by J. Kigami, developed by R.S. Strichartz, K.-S.Lau, J.X.Hu, et al.
2. The idea is proposed by H. Triebel, developed by M. Zähle, D.C. Yang, et al.
3. The idea is proposed by B.B.Mnadelbrot, developed by F. Tatom, M. Zähle, K.Yao, et al.
4. The diea is proposed by the School of Harmonic-Fractal Analysis in Nanjing University, developed by members in the School.

1. The idea by J.Kigami

Before the idea of J.Kigami, a lot of physicists paid their attention to analytic structures of a fractal set, and studied the Brownian motions on fractals, as well as proved the existence of Brownian motions on a Sierpinski gasket, such as, M.Barlow, E.Perkins, T.Lindstrom and R.Bass.

Kigami introduces the Dirichelt forms, Laplacians, heat kernels on self-similar sets, …, then show a series theorems and properties, and establish the theory of partial differential equations on fractals. He has published the paper "Harmonic calculus on the Sierpindki spaces" in 1989, then about more than 20 papers are published continuously. The book " Analysis on Fractals " [1] has been published in 2001. He has devoted his best to do the research of analysis on fractals, and obtained lots of foundation works in the area.

The main contributions of Kigami:

(1) Construction of Dirichelt forms and Laplacians on p.c.f. (post-critical fractal) self-similar sets

For a self-similar set, a topological structure, a harmonic structure and Green' operator are defined, so that for fractal PDE, the preparations have been established. For example,

Derichelt form Let V be a finite set, $l(V)=\{f:\ V\to\mathbb{R}\}$ be equipped with inner product $(u,v)=\sum_{p\in V}u(p)v(p)$, $u,v\in l(V)$; A symmetric bilinear form $\varepsilon(u,v)$ on $l(V)$ is said to be a Dirichlet form on V, if it satisfies

i) $\forall u \in l(V) \Rightarrow \varepsilon(u,u) \geq 0$;

ii) $\varepsilon(u,u) = 0 \Leftrightarrow u \in l(V)$ *is constant on* V;

iii) $\forall u \in l(V) \Rightarrow \varepsilon(u,u) \geq \varepsilon(\bar{u},\bar{u}) \geq 0$ *with* $\bar{u}(p) = \begin{cases} 1, & \text{if } u(p) \geq 1 \\ u(p), & \text{if } 0 < u(p) < 1 \\ 0, & \text{if } u(p) \leq 1 \end{cases}$.

Then, he generalizes the definitions of Dirichlet forms and Laplacian on limits of networks; moreover, on the p.c.f. self-similar structures.

(2) Discussion of the spectrum theory

Including the eigen-vaules, eigen-functions, the existence and estimates of spectra for the Neumann boundary problems and the Dirichlet boundary problems; as well as the relationships between various fractal dimensions.

(3) Construction of heat kernels

Heat kernel Let b represent boundary condition, $b = N$ as the Neumann boundary condition, and $b = D$ as the Dirichlet boundary condition. The b-heat kernel $p_b(t,x,y)$ for $(t,x,y) \in (0,+\infty) \times K \times K$ with K a p.c.f. self-similar structure is defined by

$$p_b(t,x,y) = \sum_{n \geq 1} e^{-\lambda_n^b t} \varphi_n^b(x) \varphi_n^b(y),$$

where $\{\lambda_n^b\}_{n\geq 1}$ exists for b as the sequence of eigen-values with $0 \leq \lambda_1^b \leq \cdots \leq \lambda_n^b \leq \cdots$, and $\{\varphi_n^b\}_{n\geq 1}$ is a complete orthonormal system in $L^2(K,\mu)$.

Then, 1° the parabolic maximum principle is proved; 2° an asymptotic behavior of the heat kernels is reveled; 3° other interesting properties are shown.

After the foundation work of Kigami, mathematicians in the world pay their attentions to this topic. Mainly,

① **R.S.Strichartz** a series research work on fractal analysis is completed, specially, the fractal differential equations since 1998. The summary book “Differential Equations on Fractals” published in 2006 [2], exhibits abundant new results, such as, 1° electric Network interpretations; 2° normal derivatives; 3° Gauss-Green formula and Green’s function; 4° spectral asymptotic growth; 5° integrals involving eigenfunctions; 6° conformal energy and energy measures on Sierpinski gaskets; 7° spectral decimation on some hierarchical gaskets; 8° resolvent kernel for p.c.f. self-similar fractals; and so on. R.S.Strichartz has excellent jobs for differential equations on fractals, and he has started to study the fractal differential equations on non-self-similar fractals.

② **K.-S.Lau** a series nice work for the Laplacian on p.c.f. self-similar sets, his main jobs concentrate on the harmonic structures of fractals. Recently, boundary theory on some trees and Martin boundary, as well as exit space on the Sierpinski gaskets are studied. With his colleagues and students, about hundreds papers have been published.

③ **J.X.Hu** he has a series work about fractal analysis, and written a foundational book "Introduction to Fractal Analysis" [3] . In the book, the developments of results of theory and applications of heat kernels are included. For instance, the estimates of various upper and lower bounds of heat kernels, there are accurate methods and nice results in the book.

④ **U.Mosco** establishes Laplacian on Heisenberg group, and so on.

By virtue of developing heat kernel theory on fractals to develop fractal PDE is a good idea .

2. The idea by H.Triebel

H.Triebel has an idea to establish a differential equation theory on fractals: by virtue of establishing function spaces on fractals to define fractal pseudo-differential operators over those function spaces, thus a theory of partial differential equations on fractals can be constructed. His book "Fractals and Spectra" [4] is published in 1997.

The main contributions of Triebel :

(1) Define the function spaces on fractals

By virtue of concept d -set to define the function spaces on fractals :

Definition 2.1 (d -set) *Let* $n \in \mathbb{N}$ *and* $\Gamma \in \mathbb{R}^n$ *, if for some* d *with* $0 \le d \le n$ *,* Γ *satisfies*: (i) *there exists a Borel measure* μ *on* $\mathbb{R}^n$ *such that* $\operatorname{supp} \mu = \Gamma$; (ii) *there exist two constants* $c_1 > 0$, $c_2 > 0$ *, such that* $c_1 r^d \le \mu\left(B(\gamma, r) \cap \Gamma\right) \le c_2 r^d$ *for all* $\gamma \in \Gamma$ *and* $0 < r < 1$ *. Then* $\Gamma \in \mathbb{R}^n$ *is said to be a* d *-set .*

Definition 2.2 (B-type space on d -set **)** *The B-type space* $B_{p,q}^{s,\Gamma}\left(\mathbb{R}^n\right)$ *on a* d *-set is defined as*

$$B_{p,q}^{s,\Gamma}\left(\mathbb{R}^n\right) = \left\{ f \in B_{p,q}^{s}\left(\mathbb{R}^n\right) :\ \langle f, \varphi \rangle = 0 \ \ for\ \varphi \in \mathbb{S}\left(\mathbb{R}^n\right) \ with\ \ \varphi|_{\Gamma} = 0 \right\}.$$

(2) Define the distribution dimension of a fractal

Definition 2.3 (Distribution dimension) *For a non empty Borel* d *-set* Γ *with the Lebesgue measure* $|\Gamma| = 0$ *is defined as*

$$\dim_D \Gamma = \sup\left\{ d :\ \mathfrak{C}^{-n+d,\Lambda}\left(\mathbb{R}^n\right) \ is\ non\ trivial\ \ for\ a\ compact\ \Lambda \subset \Gamma \right\}.$$

And he has a very significant result **:** $\dim_D \Gamma = \dim_H \Gamma$ **.** Thus one has an explanation : the distribution dimension is an analytical expression of the Hausdorff dimension. By this result, Triebel proves more properties of the Hausdorff dimensions of fractals.

(3) Define fractal pseudo-differential operators on some function spaces

1° **Operators on fractals** *Let* $\Gamma \in \mathbb{R}^n$ *be a compact* d *-set with* $0 < d < n$; A_s *is* *a self-adjoint operator on a Hilbert space* $H^s(\Gamma)$, $s>0$, *with* $\operatorname{dom}\left(A_s^{\frac{-1}{2}}\right)=H^s(\Gamma)$.

Then an operator on a fractal is defined as

$$\mathrm{B}=b_2(\gamma)A_s^{\frac{-1}{2}}b_1(\gamma), \qquad \gamma\in\Gamma, \quad b_j\in L^{r_j}(\Gamma), \quad j=1,2.$$

2° **Operators with fractal coefficients** *Let* $\Gamma\in\mathbb{R}^n$, *then an operator with fractal coefficients is defined as*

$$\mathrm{B}=c(x)b(x,D), \qquad x\in\Gamma, \quad c\in B^s_{r\,q}(\mathbb{R}^n), \quad j=1,2,$$

and $\operatorname{supp} c(x)$ *is compact,* $s<0$、$b(x,D)\in\Psi^\kappa_{1,\rho}(\mathbb{R}^n)$ *with* $\kappa>0$, $0\le\rho\le 1$, *and* $\Psi^\kappa_{1,\rho}(\mathbb{R}^n)$ *is the Hörmander class of pseudo-differential operators, and for* $f\in\mathbb{S}^*(\mathbb{R}^n)$

$$b(x,D)f(x)=\left(\sqrt{2\pi}\right)^{-n}\int_{\mathbb{R}^{2n}}e^{i(x-y)}b(x,\xi)f(y)dyd\xi.$$

3° **Fractal pseudo-differential operator** *Let* $\Gamma\in\mathbb{R}^n$ *be a compact* d *-set with* $0<d<n$; *take* $b(x,D)$ *as in* 2°. *Then a fractal pseudo-differential operator is defined as*

$$\mathrm{B}=b_1(\gamma)b(\gamma,D)b_2(\gamma), \qquad \gamma\in\Gamma, \quad b_j\in L^{r_j}(\Gamma), \quad j=1,2.$$

(4) Find the spectra of the above operators

By decomposition method of the operators, the spectrum theorems are obtained.

Theorem 2.1 *Let* $\Gamma\in\mathbb{R}^n$ *be a compact* d *-set with* $0<d<n$, *and* $1\le p\le+\infty$, $1\le r_1,r_2\le+\infty$, $\kappa>0$, $0\le\rho\le 1$ *with* $\frac{1}{r_2}<\frac{1}{p}<1-\frac{1}{r_1}$, *and* $\kappa-n+d>d\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$;

Then the fractal pseudo-differential operator

$$\mathrm{B}=b_1(\gamma)b(\gamma,D)b_2(\gamma), \qquad \gamma\in\Gamma, \quad b_j\in L^{r_j}(\Gamma), \quad j=1,2,$$

is a compact one on $L^p(\Gamma)$, *and its non-vanishing eigen-values* μ_k, $k=1,2,\cdots$, *repeated according the algebraic multiplicity, and ordered so that* $|\mu_1|\ge|\mu_2|\ge\cdots$ *satisfying*

$$|\mu_k|\le c\|b_1\|_{L^{r_1}(\Gamma)}\|b_2\|_{L^{r_2}(\Gamma)}k^{-\frac{\kappa-n+d}{d}}, \qquad k\in\mathbb{N},$$

where $k^{-\frac{\kappa-n+d}{d}}$ *is a fractal defect factor.*

Several similar theorems are obtained. The method in the proof is very technique and talent.

(5) Applications to other type differential equations

By using the above results, Triebel has studied : 1° fractal drums; 2° Schrodinger operators with fractal potentials; 3° non-linear elliptic equations related to fractals. And so on.

H.Triebel has contributed a talent idea, and proved a series excellent research results to the fractal differential equations.

Then, mainly, M. Zähle [6], D.C.Yang, et al study some function spaces on a fractal; some spectra of fractal drums and fractal PDE; Spectra of some operators with fractal potentials.

By virtue of establishing function spaces on fractals to develop fractal PDE is a good idea.

3. The idea by B.B.Mandelbrot

B.B.Mandelbrot, V.Ness proposed the idea: by virtue of the fractional calculus to describe some fractal behaviors of motions, to look for solutions of fractional differential equations.

The main contributions by B.B.Mandelbret and J.W.Van Ness:

They generalize the Brownian motion function $B(t)$ to the fractional Brownian motion function $B_H(t)$ by the fractional calculus in 1968 :

$$B_H(t) = \frac{1}{\Gamma\left(H + 2^{-1}\right)} \left\{ \int_{-\infty}^{t} (t-s)^{\left(H+2^{-1}\right)-1} dB(s) - \int_{-\infty}^{0} (t-s)^{\left(H+2^{-1}\right)-1} dB(s) \right\}$$

with the Hurst exponent index H, $dB(s) = Wds$, W is the white noise, $\int_{-\infty}^{t} \cdots$ is the Weyl fractional integral.

① **M.V.Berry** in the study of diffractals（waves which have encountered fractals）, he derived the relationship between the fractal dimension D_f of a fractal process and the decay exponent α of the spectrum of the process as $D_f = \dfrac{5-\alpha}{2}$ in 1979.

② **L.F.Burlaga and L.W.Kiein** study of fractal structure of the interplanetary magnetic field，they have result: the spectral decay exponent α and Hurst exponent H are related according to the relation $\alpha = 2H + 1$ in 1986.

③ **F.B.Tatom** his study of the simulation of turbulence by means of fractional calculus was in 1989, and developed a fractional differential equation involving white noise

$$\frac{d^q z}{dt^q} = \sqrt{B}\,\frac{d^g W}{dt^g} \exp\left(\omega_0 t\right)$$

with $z = y\exp\left(\omega_0 t\right)$, $y =$ output time series, $\omega_0 =$ frequency of the energy-containing eddies and $W =$ white noise, $B =$ scaling constant; q, p are positive real with $q - p = \dfrac{\alpha}{2}$.

④ **A.L.Mchaute** published a book "Fractal Geometries — Theory and Applications" in which he defines derivatives by a non-integral order of a function, and spectral analysis in non-integral derivatives as well as fractal geometries.

⑤ **M. Zähle, K.Yao et al** they have published a series papers about relations between fractional orders and fractal dimensions, they try to consider the fractal dimensions of solutions of fractional order differential equations, and find a general rule.

All results above make an idea of physicists, they believe that the fractional calculus and the fractal dimensions of all motions have very closed relations. They use the fractional calculus and fractional differential equations to the study of fractals and random fractals.

By virtue of the fractional calculus to develop fractal PDE is a good idea .

4. The idea by the School of Harmonic-Fractal Analysis in NJU

The idea of the School in NJU: to establish "the fractal calculus" for describing a rate of change (velocity, or speed) of a fractal because the Newton calculus does not work for fractals, then the fractal analysis, fractal PDE, fractal dynamics, … can be developed.

We do the following contributions:

(1) Find a suitable underlying space that fractal sets live in —— local fields are suitable for laying fractal sets

A local field K_p is a topological field, it is locally compact, totally disconnected, complete metric space with non-archimedean norm $|x|$, addition $\oplus$, multiplication $\otimes$, where $p \geq 2$ is a prime, thus $K_p \equiv \left(K_p, \oplus, \otimes, |x|\right)$. K_p is said to be a local field.

For a prime $p \geq 2$, $\forall x \in K_p$ can be expressed as a series

$$x = x_{-s}\beta^{-s} + x_{-s+1}\beta^{-s-1} + \cdots + x_{-1}\beta^{-1} + x_0\beta^0 + x_1\beta^1 + x_2\beta^2 + \cdots$$

with $x_j \in \{0,1,\cdots,p-1\}$, $j = -s, -s+1, \cdots$, $s \in \mathbb{Z}$. $\beta \in K_p$ is the prime element of K_p with non-archimedean norm $|\beta| = p^{-1}$, and the addition $\oplus$, multiplication $\otimes$ is term by term mod p [9]. The dyadic group with $p = 2$ used in the computer science is a special local field.

(2) Establish the Harmonic analysis on local fields

A series work of harmonic analysis on local fields is established, including: ① the character group theory on the character group Γ_p of a local field K_p; ② the Fourier analysis on local field K_p; ③ the distribution theory on the distribution space $\mathbb{S}^*\left(K_p\right)$ (Schwartz distribution type space) of the test function class $\mathbb{S}\left(K_p\right)$ (Schwartz type space) of K_p; ④ the operator theory, specially, the pseudo-differential operator theory; ⑤ the principle to establish a new type calculus is proposed.

(3) Define the p-type calculus on local fields

For a Haar measurable function $f:\ K_p \to \mathbb{C}$, we define the p-type derivative, p-type integral of f by virtue of the pseudo-differential operator.

Definition 4.1 (point-wise p-type derivative and L^r strong p-type derivative) *Let $\alpha > 0$, if for a complex valued Haar measurable function $f: K_p \to \mathbb{C}$ on K_p, the integral*

$$T_\alpha f(x) \equiv \int_{\Gamma_p} \left\{ \int_{K_p} \langle\xi\rangle^{\alpha} f(t) \overline{\chi_\xi}(t-x)\, dt \right\} d\xi$$

*exists at $x \in K_p$, then $T_\alpha f(x)$ is called **a point-wise α-order p-type derivative of** $f(x)$ **at** x, denoted by $f^{\langle\alpha\rangle}(x)$.*

Let $f_k(x) = \begin{cases} f(x), & |x| \le p^k \\ 0, & |x| > p^k \end{cases}$, $k \in \mathbb{Z}$. *If there exists $g \in L^r(K_p)$, $1 \le r < +\infty$, such that $\lim\limits_{k\to+\infty} \|g - T_\alpha f_k\|_{L^r(K_p)} = 0$, then g is called **an L^r strong α-order p-type derivative of** $f(x)$, denoted by $g = D^{\langle\alpha\rangle} f$.*

Definition 4.2 (point-wise p-type integral and L^r strong p-type integral) *Let $\alpha > 0$, if for a complex valued Haar measurable function $f: K_p \to \mathbb{C}$ on K_p, the integral*

$$T_{-\alpha} f(x) \equiv \int_{\Gamma_p} \left\{ \int_{K_p} \langle\xi\rangle^{-\alpha} f(t) \overline{\chi_\xi}(t-x)\, dt \right\} d\xi$$

*exists at $x \in K_p$, then $T_\alpha f(x)$ is called **a point-wise α-order p-type integral of** $f(x)$ **at** x, denoted by $f_{\langle\alpha\rangle}(x)$.*

*If there exists $h \in L^r(K_p)$, $1 \le r < +\infty$, such that $\lim\limits_{k\to+\infty} \|h - T_{-\alpha} f_k\|_{L^r(K_p)} = 0$, then h is called **an L^r strong α-order p-type integral of** $f(x)$, denoted by $h = I_{\langle\alpha\rangle} f$.*

We mention that: This type calculus was proposed firstly by J.E.Gibbs in 1969 [7], called the Gibbs derivative, or logical derivative. And many mathematicians in the world have contributed excellent work, such as, P.L.Butzer [8], R.S. Stankovic [12], C.W.Onneweer [12]. However, they all defined the "Gibbs derivative" by infinite series forms. The school of Harmonic-Fractal analysis in NJU has more contributions since 1970. The principle to establish a new calculus is proposed by them, and define the p-type calculus by virtue of the pseudo-differential operators; moreover, lots of properties of the p-type calculus are proved.

（4） Study function spaces on local fields

We use the p-type calculus to establish and study function spaces, such as, Hölder spaces, Lipschitz spaces, …. And we find some function spaces in which the p-type differentiable functions live, for example,

Theorem 4.1 *The Hölder type space* $C^{\alpha}\left(K_p\right)$, $\alpha \in [0,+\infty)$ *has the properties*

(i) *if* $f \in C^{\alpha}\left(K_p\right)$, *the the function* f *has the* p-*type* λ-*order derivative* $T_{\langle\cdot\rangle^{\lambda}} f(x)$ *at* $x \in K_p$, *and* $T_{\langle\cdot\rangle^{\lambda}} f(x) \in C^{\alpha-\lambda}\left(K_p\right)$ *for* $0 \le \lambda \le \alpha$;

(ii) *if* $T_{\langle\cdot\rangle^{\alpha}} f(x) \in C^{0}\left(K_p\right)$, *then for* $0 \le \lambda \le \alpha$, *the function* f *has the* p-*type derivative* $T_{\langle\cdot\rangle^{\alpha-\lambda}} f(x)$, $x \in K_p$, *and* $T_{\langle\cdot\rangle^{\alpha-\lambda}} f(x) \in C^{\lambda}\left(K_p\right)$.

This theorem shows that: Functions which have p-type derivatives live in the Hölder spaces: $f \in C^{\alpha}\left(K_p\right) \rightleftarrows \forall m \le \alpha,\ f^{\langle m\rangle}$ *continuous*.

Note that: this property is quite different with that of $\mathbb{R}^n$ case, since in case of $\mathbb{R}^n$, for the Hölder space $C^{\alpha}\left(\mathbb{R}^n\right)$, $\alpha \in [0,+\infty)\setminus\mathbb{N}$, there are gaps at the integers without definitions.

Theorem 4.2 *For a local field* K_p, *we have* $\mathrm{Lip}\left(\alpha, K_p\right) = C^{\alpha}\left(K_p\right)$, $\alpha \in (0,+\infty)$.

This theorem shows that: The α-Lip class may take $\alpha \in (0,+\infty)$, and is equivalent to the Hölder space. So that this is quite different with that of $\mathbb{R}^n$ case.

Theorem 4.3 *For the Weierstrass type function*

$$W(x)=\sum_{k=1}^{+\infty} p^{(s-2)\,k}\,\mathrm{Re}\,\chi\left(\beta^{-k}x\right),\qquad 1\le s<2,\quad x=\sum_{j=0}^{+\infty} x_j\beta^{j}\in D,$$

it is infinite p - *type integrable, and* m - *order*（$m<2-s$）p-*type differentiable*; *Moreover,*

$$W^{\langle m\rangle}(x)=\sum_{k=1}^{+\infty} p^{(s+m-2)\,k}\,\mathrm{Re}\,\chi\left(\beta^{-k}x\right),\qquad x\in D.$$

Theorem 4.4 *For the Weierstrass type function*

$$W(x)=\sum_{k=1}^{+\infty} p^{(s-2)\,k}\,\mathrm{Re}\,\chi\left(\beta^{-k}x\right),\qquad 1\le s<2,\qquad x\in D$$

(i) *if* $p=2$, *then* $\dim_H \mathbb{G}\left(W(x), D\right)=s$, *a.e.* $s\in(1,2)$;

（ⅱ） *if* $p>2$ *then* $\dim_H \mathbb{G}\left(W\left(x\right),D\right)=s$, *a.e.* $s\in\left(\log_p\left(2p-1\right),2+\log_p y\left(b_p\right)\right)$

with $b_p=1-\cos\dfrac{\left(p-1\right)\pi}{p}\left(1-\cos\dfrac{2\pi}{p}\right)^{-1}$, *and* $y\left(b_p\right)$ *is the smallest positive number satisfying*

$g\left(y\left(b_p\right)\right)=g'\left(y\left(b_p\right)\right)=0$, *where* $g\left(x\right)$ *is a power series with coefficients* $g_j\in\left[-b,b\right]$, i.e.,

$g\left(x\right)=\sum\limits_{j=0}^{+\infty}g_jx^j$ *exists*.

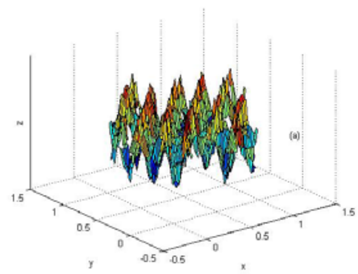
This theorem shows that : There are closed relationship between the p - type calculus and the dimensions of a fractal.

（5） Establish fractal PDE on local fields

① **partial differential equation with fractal boundary**

$$\begin{cases}\dfrac{\partial^2u}{\partial t^2}=\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}, & t\in\mathbb{R},\ (x,y)\in D,\\ u\big|_{t=0}=\varphi(x,y), & (x,y)\in D,\\ \left.\dfrac{\partial u}{\partial t}\right|_{t=0}=\psi(x,y), & (x,y)\in D,\\ u\big|_{\gamma}=0, & t\in\mathbb{R}\ ,\end{cases} \qquad \textbf{(*)}$$

where γ is a fractal. Then we have a fractal solution under some conditions. This result is valuable for the studying about partial differential equations since it shows the case which can not be solved in the classical PDE. The following graph

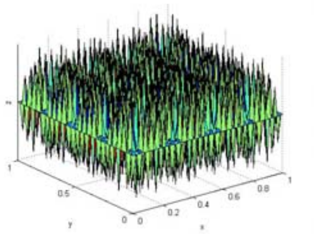


is an approximation solution of problem (*).

Also, the *p*-type PDE with fractal boundary is considered

$$\begin{cases}\dfrac{\partial^{\langle 2\rangle}u}{\partial t^{\langle 2\rangle}}=\dfrac{\partial^{\langle 2\rangle}u}{\partial x^{\langle 2\rangle}}+\dfrac{\partial^{\langle 2\rangle}u}{\partial y^{\langle 2\rangle}}, & t\in\mathbb{R},\ (x,y)\in D,\\ u\big|_{t=0}=\varphi(x,y), & (x,y)\in D,\\ \left.\dfrac{\partial^{\langle 1\rangle}u}{\partial t^{\langle 1\rangle}}\right|_{t=0}=\psi(x,y), & (x,y)\in D,\\ u\big|_{\gamma}=0, & t\in\mathbb{R}\ .\end{cases} \qquad \textbf{(**)}$$

The following graph



is an approximate solution of problem (**).

② **the kernel of operator** $T^{\alpha} \equiv T_{\langle\cdot\rangle^{\alpha}}$ ——

For the *p*-type calculus operator T^{α}, we have $T^{\alpha}\varphi = \left(\langle\xi\rangle^{\alpha}\varphi^{\wedge}\right)^{\vee}$, $\varphi \in \mathbb{S}(K_p)$**;** and

$$\left\langle T^{\alpha} f, \varphi\right\rangle = \left\langle f, T^{\alpha}\varphi\right\rangle, \qquad f \in \mathbb{S}^*(K_p).$$

Hua Qiu proves that the kernel of T^{α} is the distribution $\kappa_{\alpha} \in \mathbb{S}^*(K_p)$, where the space $\mathbb{S}^*(K_p)$ is the distribution space of Schwartz type space $\mathbb{S}(K_p)$ on a local field K_p, and for a distribution $\pi_{\alpha} \in \mathbb{S}^*(K_p)$ with $(\pi_{\alpha}, \varphi) = \int_K |x|^{\alpha-1}(\varphi(x) - \varphi(0))dx$, $\varphi \in \mathbb{S}(K_p)$, it holds

$$\kappa_{\alpha} = \begin{cases} \left\{\dfrac{1-p^{\alpha}}{1-p^{-\alpha-1}}\pi_{-\alpha} - \dfrac{1-p^{\alpha}}{1-p^{\alpha+1}}\right\}\Delta_0, & \alpha \neq 0, -1 \\ \delta, & \alpha = 0 \\ \left(1-\dfrac{1}{p}\right)\left\{1-\log_p|x|\right\}\Delta_0, & \alpha = -1 \end{cases}, \quad \forall \alpha \in \mathbb{R}\setminus\{0\}.$$

This result gives an excellent contribution to study the PDE on local fields.

③ **applications to medical science** ——

(a) malignant cases of liver's cancer

To estimate the Hausdorff dimension or box dimension of a Liver's cancer block, then determine the malignant cases.

(b) **APOLT（auxiliary partial orthotopic liver's transplantation）**

Use some mathematical models to evaluate blood flow, blood volume, …, to determine how much volumes to transplant ?

(c) **find what genes control the liver' cancers**

Use track records in CMOS chips to analyze the effects of each gene to liver's cancers, and then may know the effects of genes.

（6） some open problems

① Applications of Function spaces on K_p

We have defined Hölder space、Sobolev spaces、Lebesgue spaces、Triebel B-type and F-type spaces. As we know, these spaces in the classical cases play very important roles in the applications, specially, in the PDE theory and applications. However, we have a few bits of results in applications of function spaces over local fields.

So that it is worthwhile to consider defining new spaces over local fields, and applying function spaces to fractal PDE theory, and physical problems, … .

② Fractal analysis on K_p

p -type calculus is quite suitable for studying the fractal problems, because so called fractal, one of its features is that it has no classical derivatives, then the classical calculus is failed to deal with rate of change of fractal motion. However, the *p*-type calculus is very valued to deal with the fractal problems.

Specially, in the fractal dynamics、fractal ordinary differential equations、partial differential equations、…, there are lots of open problems in the fractal analysis worthwhile to study.

③ Fourier analysis on the multiplication group $\left(K^{*},\otimes,|x|\right)$

We may develop the Fourier theory on the multiplication group $\left(K^{*},\otimes\right)$ —— Mellin transformation theory. Since the multiplication group is not self-dual, so that the Fourier analysis on $\left(K^{*},\otimes,|x|\right)$ must be quite different, and must have a lot of interesting results, so develop harmonic analysis and fractal analysis on $\left(K^{*},\otimes,|x|\right)$ not only have the theoretical senses, but also have application senses. For example, the Mellin transforms, the Riesz transforms and the Hanker transforms, and so on.

In the last century, 1975, Taibleson published a book “Fourier Analysis on Local fields” [9] it is a fundamental one for the harmonic analysis on local fields. Then, lots of mathematicians in all of the world pay their great attention to the study of this topic, and have got more and more excellent results in this area. From 2007—2009[10,11], there are 3 times International Conferences held in Europe (Serbia, Greece, and Germeny).

Chinese mathematicians also have nice contributions to the study. Recently, we have a book "Harmonic Analysis and Fractal Analysis on Local fields with Applications" [12] published by Science Publishing Company, Beijing, China. In this book, some basic concepts and new results of the recent development are included in. We hope it may gives a little help for those scholars who are interested in the topics.

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