Iterated function systems with a given continuous stationary distribution

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Notation

IFS with probabilities $\{\mathbb{R}^d; f_i, p_i,\ i = 1, ..., n\}$

$f_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, ..., n$, are functions,
p_i are associated non-negative numbers with $\sum_{i=1}^n p_i = 1$.

The address map

If the maps $f_i : \mathbb{R}^d \to \mathbb{R}^d$ are contractions, i.e. if there exists a constant $c < 1$ such that $|f_i(x) - f_i(y)| \leq c|x - y|$, for all $x, y \in \mathbb{R}^d$, then the limits

$$\widehat{Z}(\mathbf{i}) = \lim_{k \to \infty} f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(x),$$

exist for any $\mathbf{i} = i_1 i_2 i_3 ... \in \{1, ..., n\}^{\mathbb{N}}$,
(with limits independent of $x \in \mathbb{R}^d$).

Set and measure attractors

In particular it then follows that

- the set

$$A := \{\widehat{Z}(\mathbf{i}), \mathbf{i} \in \{1, ..., n\}^{\mathbb{N}}\}$$

is the unique nonempty compact set A satisfying

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• the measure $\mu(\cdot) := P(\mathbf{i}; \widehat{Z}(\mathbf{i}) \in \cdot)$ is the unique probability measure μ, supported on A, satisfying the invariance equation

$$\mu(\cdot) = \sum_{i=1}^{n} p_i \mu(f_i^{-1}(\cdot)).$$

(the measure-attractor)

Invariant measures/stationary measures

The measure-attractor, μ, is the unique stationary distribution of the Markov chain $\{X_k\}$ obtained by random (independent) iterations with the functions, f_i, chosen with the corresponding probabilities, p_i, i.e. μ is the unique probability measure with the property that if X_0 is μ-distributed and we recursively define

$$X_{k+1} = f_{I_{k+1}}(X_k),$$

where $\{I_k\}$ is a sequence of independent random variables with $P(I_k = i) = p_i$, then $\{X_k\}$ will be a (strictly) stationary process.

Sufficient average contraction conditions for a.s. convergence of reversed iterates/existence of address map

$$E\mathbf{d}(f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(x), f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(y)) \leq c\mathbf{d}(x, y), \qquad (1)$$

for all x, y, for some $c < 1$, $k \geq 1$, and metric $\mathbf{d}$.

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If for instance, $d = 1$, and we use $\mathbf{d}(x, y) = \int_x^y \phi(t)dt$, where $\phi(x) = E|\frac{d}{dx} f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k})(x)|$, then if

$$E|\left(\frac{d}{dx} f_{i_1} \circ f_{i_2} \cdots \circ f_{i_{m+1}}\right)(x)| \leq cE|\left(\frac{d}{dx} f_{i_1} \circ f_{i_2} \cdots \circ f_{i_m}\right)(x)|,$$

for any x, for some $m \geq 0$, then (1) holds (with $k = 1$).

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(Applications: Image coding, simulations, parametrisations of probability distributions, useful theoretical representations.)

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Properties
$F^{-1}(F(x)) \leq x$ and $F(F^{-1}(u)) \geq u$. (If μ is continuous, i.e. if F is continuous then $F(F^{-1}(u)) = u$, for $0 < u < 1$.)

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Therefore, if $U \in U(0,1)$, the uniform distribution on the unit interval, then

$$P(F^{-1}(U) \leq x) = P(U \leq F(x)) = F(x),$$

i.e. $F^{-1}(U)$ is a μ-distributed random variable.

A simple solution to the inverse problem for continuous probability measures on $\mathbb{R}$.

Theorem

A continuous distribution, μ, on $\mathbb{R}$ with distribution function, F, is the measure-attractor of the IFS with monotone maps

$$f_i(x) := F^{-1} \circ u_i \circ F(x),$$

and probabilities $p_i = 1/n$, where $u_i(u) = u/n + (i-1)/n$, $0 \leq u \leq 1$, $i = 1, 2, ..., n$.

Proof

If $\widehat{Z}^F$ denotes the limit of the reversed iterates of the system with f_i chosen with probability $1/n$, (and $\widehat{Z}^U$ denotes the corresponding for the IFS with maps u_i) then

$$\begin{aligned}
\widehat{Z}^F &:= \lim_{k\to\infty} \widehat{Z}_k^F(x) := \lim_{k\to\infty} f_{I_1} \circ \cdots \circ f_{I_k}(x) \\
&= \lim_{k\to\infty} F^{-1} \circ u_{I_1} \circ F \circ F^{-1} \circ u_{I_2} \circ F \circ F^{-1} \circ u_{I_k} \circ F(x) \\
&= \lim_{k\to\infty} F^{-1} \widehat{Z}_k^U(F(x)) = F^{-1}(\widehat{Z}^U) \quad a.s.
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\end{aligned}$$

From the above it follows that

$$P(\widehat{Z}^F \le y) = P(F^{-1}(\widehat{Z}^U) \le y) = P(\widehat{Z}^U \le F(y)) = F(y).$$

Note

If μ is a continuous probability measure being the measure-attractor of $\{\mathbb{R}; f_i, p_i,\ i = 1, ..., n\}$, with $p_i \neq 1/n$ for some n, then there exists another IFSp (non-overlapping, with uniform probabilities) having μ as its measure-attractor.

Example

Let μ be the probability measure with triangular density function

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Then μ is the unique stationary distribution of the Markov chain generated by random iteration with the functions

$$f_1(x) = \begin{cases} \frac{x}{\sqrt{2}} & 0 \le x \le 1 \\ \sqrt{2x - \frac{x^2}{2} - 1} & 1 \le x \le 2, \end{cases}$$

and

$$f_2(x) = \begin{cases} 2 - \sqrt{1 - \frac{x^2}{2}} & 0 \le x \le 1 \\ 2 - \sqrt{2 - 2x + \frac{x^2}{2}} & 1 \le x \le 2, \end{cases},$$

chosen uniformly at random.

Histograms of the empirical distribution along a trajectory of a Markov chain having the triangular distribution as its unique stationary distribution.

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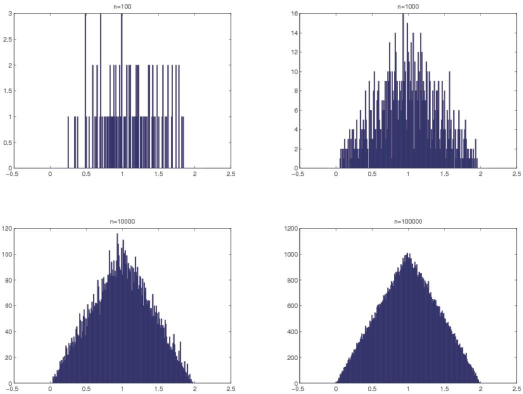

Example: Exponential distributions (upper figures), 1-variable mixtures of these exponential distributions (lower figures)