Projections of Mandelbrot percolations

Michał Rams[1] Károly Simon[2]

[1] Institute of Mathematics
Polish Academy of Sciences
Warsaw, Poland
http://www.impan.pl/~rams/

[2] Department of Stochastics
Institute of Mathematics
Technical University of Budapest
www.math.bme.hu/~simonk

12 December 2012
Hong Kong

Outline

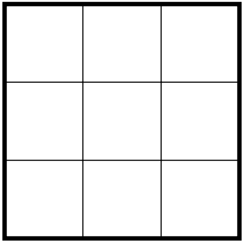
All new results are joint with Michal Rams, Warsaw IMPAN

Michal visited me last week in Budapest and while we were preparing our joint talk, he got a terrible flu which prevented him from participating in this conference.

Outline

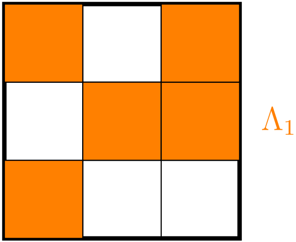
Fractal percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1-p$. In the squares retained after the previous step we repeat the same process at infinitum.



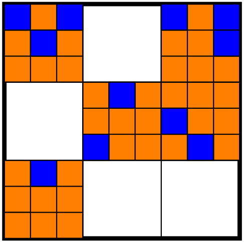
Fractal percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1 - p$. In the squares retained after the previous step we repeat the same process at infinitum.



Fractal percolation, introduced by Mandelbrot early 1970's:

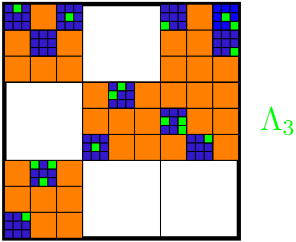
We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1-p$. In the squares retained after the previous step we repeat the same process at infinitum.



Λ_2

Fractal percolation, introduced by Mandelbrot early 1970's:

We partition the unit square into M^2 congruent sub squares each of them are independently retained with probability p and discarded with probability $1-p$. In the squares retained after the previous step we repeat the same process at infinitum.



Let Λ_n be the union of the level n retained squares. Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

It was proved by Falconer and independently Mauldin, Willims that conditioned on non-extinction:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The expected number of descendants of every square is: $M^2 \cdot p$. Therefore, if $M^2 \cdot p < 1$ then $\Lambda = \emptyset$ a.s.

Let Λ_n be the union of the level n retained squares. Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

It was proved by Falconer and independently Mauldin, Willims that conditioned on non-extinction:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The expected number of descendants of every square is: $M^2 \cdot p$. Therefore, if $M^2 \cdot p < 1$ then $\Lambda = \emptyset$ a.s.

Let Λ_n be the union of the level n retained squares. Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

It was proved by Falconer and independently Mauldin, Willims that conditioned on non-extinction:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The expected number of descendants of every square is: $M^2 \cdot p$. Therefore, if $M^2 \cdot p < 1$ then $\Lambda = \emptyset$ a.s.

Let Λ_n be the union of the level n retained squares. Then the statistically self-similar set of interest is:

$$\Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

It was proved by Falconer and independently Mauldin, Willims that conditioned on non-extinction:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

The expected number of descendants of every square is: $M^2 \cdot p$. Therefore, if $M^2 \cdot p < 1$ then $\Lambda = \emptyset$ a.s.

So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either

  (a) $\Lambda = \emptyset$ or
  (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \le 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_{\mathrm{H}}(\Lambda) < 1$ (but $\Lambda \ne \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either

  (a) $\Lambda = \emptyset$ or

  (b) $\dim_{\mathrm{H}}(\Lambda) > 1$.

Recall:

$$\dim_{\mathrm{H}} \Lambda = \dim_{\mathrm{B}} \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \leq 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

So, we have almost surely:

- If $p \le 1/M^2$ then $\Lambda = \emptyset$.
- If $1/M^2 < p < 1/M$ then $\dim_H(\Lambda) < 1$ (but $\Lambda \neq \emptyset$ with positive probability).
- If $p > \frac{1}{M}$ then either
  - (a) $\Lambda = \emptyset$ or
  - (b) $\dim_H(\Lambda) > 1$.

Recall:

$$\dim_H \Lambda = \dim_B \Lambda = \frac{\log(M^2 \cdot p)}{\log M} \text{ a.s.}$$

Marstrand Theorem

Theorem (Marstrand)

Let $B \subset \mathbb{R}^2$ be a Borel set.

1. *If* $\dim_H(B) \le 1$ *then for* $\mathcal{L}$eb*-a.e.* θ*, we have*

$$\dim_H(\text{proj}_\theta(B)) = \dim_H(B)$$

2. *If* $\dim_H(B) > 1$ *then for* $\mathcal{L}$eb*-a.e.* θ*, we have*

$$\mathcal{L}\text{eb}\,(\text{proj}_\theta(B)) > 0.$$

Marstrand Theorem

Theorem (Marstrand)

Let $B \subset \mathbb{R}^2$ *be a Borel set.*

1. *If* $\dim_{\mathrm{H}}(B) \leq 1$ *then for* $\mathcal{L}\mathrm{eb}$*-a.e.* θ*, we have*

$$\dim_{\mathrm{H}}(\mathrm{proj}_\theta(B)) = \dim_{\mathrm{H}}(B)$$

2. *If* $\dim_{\mathrm{H}}(B) > 1$ *then for* $\mathcal{L}\mathrm{eb}$*-a.e.* θ*, we have*

$$\mathcal{L}\mathrm{eb}\left(\mathrm{proj}_\theta(B)\right) > 0.$$

Marstrand Theorem

Theorem (Marstrand)

Let $B \subset \mathbb{R}^2$ *be a Borel set.*

1. *If* $\dim_{\mathrm{H}}(B) \leq 1$ *then for* $\mathcal{L}\mathrm{eb}$*-a.e.* θ, *we have*

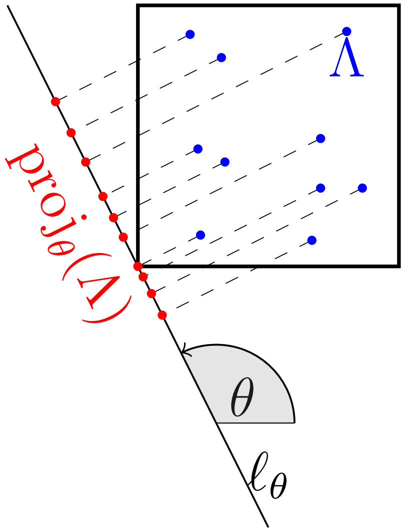
$$\dim_{\mathrm{H}}(\mathrm{proj}_\theta(B)) = \dim_{\mathrm{H}}(B)$$

2. *If* $\dim_{\mathrm{H}}(B) > 1$ *then for* $\mathcal{L}\mathrm{eb}$*-a.e.* θ, *we have*

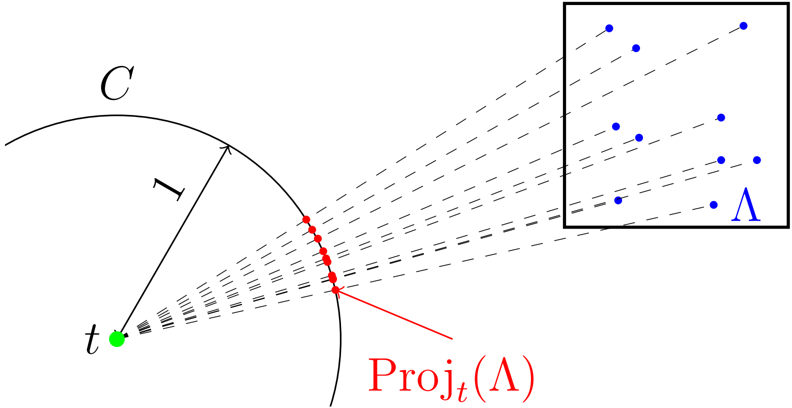
$$\mathcal{L}\mathrm{eb}\left(\mathrm{proj}_\theta(B)\right) > 0.$$

Outline

Orthogonal projection to ℓ_θ

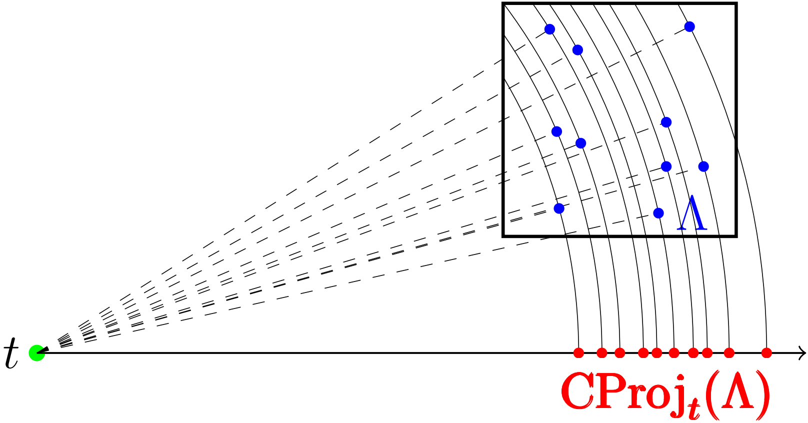


Radial and co-radial projections with center t



Let $\text{CProj}_t(\Lambda) := \{\text{dist}(t, x) : x \in \Lambda\}$ ($\text{CProj}_t(\Lambda)$ is the set of the length of dashed lines above).

The co-radial projection



Outline

Λ percolates

Let $\Lambda(\omega)$ be a realization of this random Cantor set. We say that $\Lambda(\omega)$ percolates if there is a connected component of $\Lambda(\omega)$ which connects the left and the right walls of the square $[0,1]^2$.

Let us write $E_{|\leftrightsquigarrow|}$ for the event that the random self-similar set Λ percolates.

Theorem [J.T Chayes, L. Chayes, R. Durrett] [1]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf\left\{p : \mathbb{P}_p\left(E_{|\leftrightsquigarrow|}\right) > 0\right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup\left\{p : \mathbb{P}_p\left(TD\right) = 1\right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p > p_c$ then Λ percolates with positive probability.

Theorem [J.T Chayes, L. Chayes, R. Durrett] [1]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf \left\{ p : \mathbb{P}_p \left(E_{|\leftrightsquigarrow|} \right) > 0 \right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup \left\{ p : \mathbb{P}_p \left(TD \right) = 1 \right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p > p_c$ then Λ percolates with positive probability.

Theorem [J.T Chayes, L. Chayes, R. Durrett] [1]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf \left\{ p : \mathbb{P}_p \left(E_{|\leftrightsquigarrow|} \right) > 0 \right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup \left\{ p : \mathbb{P}_p \left(TD \right) = 1 \right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p > p_c$ then Λ percolates with positive probability.

Theorem [J.T Chayes, L. Chayes, R. Durrett] [1]

Let TD be the event that Λ is totally disconnected. That is all connected components are singletons. Let

$$p_c := \inf \left\{ p : \mathbb{P}_p \left(E_{|\leftrightsquigarrow|} \right) > 0 \right\}$$

Then $0 < p_c < 1$ and

$$p_c = \sup \left\{ p : \mathbb{P}_p \left(TD \right) = 1 \right\}.$$

If $p < p_c < 1$ then all connected components of Λ are singletons. If $p > p_c$ then Λ percolates with positive probability.

Theorem (Falconer and Grimmett)

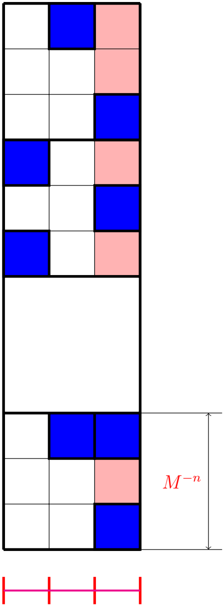
Assume that

$$p > \frac{1}{M} \tag{1}$$

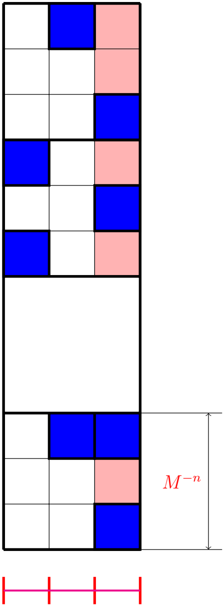
Then the orthogonal projection to the x-axis and to the y-axis of Λ contain an interval almost surely, conditioned on non-extinction.

Our research was inspired by this paper. The idea of the proof: use large deviation theory for the INDEPENDENT number of level n successors of squares which are in the same vertical column. $\dim_{\mathrm{H}} \Lambda > 1 \Longrightarrow \exists n, \exists$ a level n column with exponentially many squares. This column is the biggest column on the next figure.

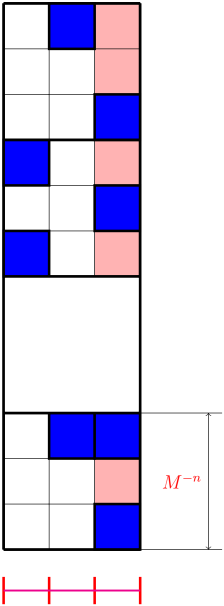
There are exponentially many level n squares in it. When we move from level n to level $n+1$ independently each of them gives birth an expected number of $pM > 1$ number of level $n+1$ squares in the red column. By large deviation th. there is a superexponentially small probability that the number of level $n+1$ squares is not more that a fixed $\alpha > 1$ multiple of the level n squares in the red column. This implies that in each column on the figure there will be $\alpha > 1$ times more squares of level $n+1$ than of level n except with a super exponentially small probability.



There are exponentially many level n squares in it. When we move from level n to level $n+1$ independently each of them gives birth an expected number of $pM > 1$ number of level $n+1$ squares in the red column. By large deviation th. there is a superexponentially small probability that the number of level $n+1$ squares is not more that a fixed $\alpha > 1$ multiple of the level n squares in the red column. This implies that in each column on the figure there will be $\alpha > 1$ times more squares of level $n+1$ than of level n except with a super exponentially small probability.



There are exponentially many level n squares in it. When we move from level n to level $n+1$ independently each of them gives birth an expected number of $pM > 1$ number of level $n+1$ squares in the red column. By large deviation th. there is a superexponentially small probability that the number of level $n+1$ squares is not more that a fixed $\alpha > 1$ multiple of the level n squares in the red column. This implies that in each column on the figure there will be $\alpha > 1$ times more squares of level $n+1$ than of level n except with a super exponentially small probability.



Outline

Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$\forall \theta \in [0, \pi]$, $\text{proj}_\theta(\Lambda)$ containes an interval .

Further,

$\forall t \in \mathbb{R}^2$, $\text{Proj}_t(\Lambda)$ and $\text{CProj}_t(\Lambda)$ contain an interval .

Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \quad \mathrm{proj}_\theta(\Lambda) \text{ containes an interval .}$$

Further,

$$\forall t \in \mathbb{R}^2, \quad \mathrm{Proj}_t(\Lambda) \text{ and } \mathrm{CProj}_t(\Lambda) \text{ contain an interval .}$$

Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ containes an interval .}$$

Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \text{ and } \mathrm{CProj}_t(\Lambda) \text{ contain an interval .}$$

Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ containes an interval .}$$

Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \text{ and } \mathrm{CProj}_t(\Lambda) \text{ contain an interval .}$$

Theorem [R., S.] (When $p > \frac{1}{M}$)

We assume that

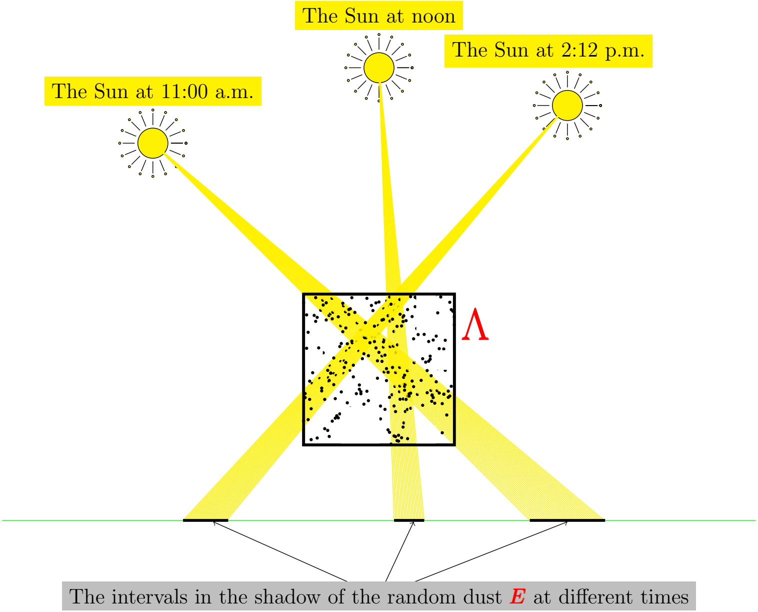
$$p > \frac{1}{M}.$$

Then the following statements hold almost surely conditioned on $\Lambda \neq \emptyset$:

$$\forall \theta \in [0, \pi], \ \mathrm{proj}_\theta(\Lambda) \text{ containes an interval .}$$

Further,

$$\forall t \in \mathbb{R}^2, \ \mathrm{Proj}_t(\Lambda) \text{ and } \mathrm{CProj}_t(\Lambda) \text{ contain an interval .}$$


The Sun at noon
The Sun at 2:12 p.m.
The Sun at 11:00 a.m.
Λ
The intervals in the shadow of the random dust E at different times

Theorem [R., S.] If $\frac{1}{M^2} < p \leq \frac{1}{M}$

Theorem

Let $\ell \subset \mathbb{R}^2$ be a straight line and let Λ_ℓ be the orthogonal projection of Λ to ℓ.

*Then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ we have:*

$$\dim_{\mathrm{H}}(\Lambda_\ell) = \dim_{\mathrm{H}}(\Lambda). \quad (2)$$

Actually much more is true:

Theorem [R., S.] If $\frac{1}{M^2} < p \leq \frac{1}{M}$

Theorem

Let $\ell \subset \mathbb{R}^2$ be a straight line and let Λ_ℓ be the orthogonal projection of Λ to ℓ.

*Then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ we have:*

$$\dim_H(\Lambda_\ell) = \dim_H(\Lambda). \quad (2)$$

Actually much more is true:

Theorem [R., S.] If $\frac{1}{M^2} < p \leq \frac{1}{M}$

Theorem

Let $\ell \subset \mathbb{R}^2$ be a straight line and let Λ_ℓ be the orthogonal projection of Λ to ℓ.

*Then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ we have:*

$$\dim_H(\Lambda_\ell) = \dim_H(\Lambda). \tag{2}$$

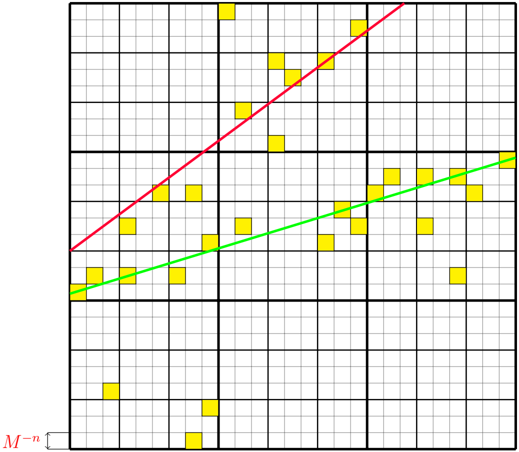
Actually much more is true:

Lines intersect $\leq c \cdot n$ squares of level n

Theorem (R., S.)

*If $\frac{1}{M^2} < p \leq \frac{1}{M}$ then for almost all realizations of Λ (conditioned on $\Lambda \neq \emptyset$) and for **all** straight lines ℓ : there exists a constant C such that **the number of level n squares having nonempty intersection with Λ is at most $c \cdot n$**.*

*On the other hand, almost surely for n big enough, we can find **some** line of 45° angle which intersects $\texttt{const} \cdot n$ level n squares.*



Recall:

$\frac{1}{M^2} < p \leq \frac{1}{M}$ $\Rightarrow$ *Then every line ℓ intersects at most $\mathtt{const} \cdot n$ level n squares.*

Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_H(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
4. If $p \geq p_c$ then Λ percolates.

Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_{\mathrm{H}}(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
4. If $p \geq p_c$ then Λ percolates.

Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_H(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
4. If $p \geq p_c$ then Λ percolates.

Summary

1. If $0 < p \leq 1/M^2$ then Λ dies out in finitely many steps almost surely.
2. If $\frac{1}{M^2} < p < \frac{1}{M}$ The $\Lambda \neq \emptyset$ with positive probability but $\dim_{\mathrm{H}}(\Lambda) = \frac{\log(M^2 p)}{M} < 1$. For almost all non-empty realizations, for all projections (all radial, co-radial and all orthogonal projections) the dimension of Λ does not decrease under the projection .
3. If $\frac{1}{M} < p < p_c$. Conditioned on non-extinction, almost surely: all projections of Λ contain some intervals but Λ is totally disconnected .
4. If $p \geq p_c$ then Λ percolates.

Definition

We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0,\ f'_y \neq 0$.

Theorem (R., S.)

If $p > \frac{1}{M}$ $(\dim_{\mathrm{H}} \Lambda > 1)$ then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x+y : (x,y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x,y) \in \Lambda\} \supset$ interval .

Definition

We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0,\ f'_y \neq 0$.

Theorem (R., S.)

If $p > \frac{1}{M}$ ($\dim_{\mathrm{H}} \Lambda > 1$) then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x + y : (x,y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x,y) \in \Lambda\} \supset$ interval .

Definition

We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0,\ f'_y \neq 0$.

Theorem (R., S.)

If $p > \frac{1}{M}$ ($\dim_{\mathrm{H}} \Lambda > 1$) then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x+y : (x,y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x,y) \in \Lambda\} \supset$ interval .

Definition

We say that $f[0,1]^2 \to \mathbb{R}$ is a strictly monotonic smooth function if $f \in \mathcal{C}^2[0,1]$ and $f'_x \neq 0,\ f'_y \neq 0$.

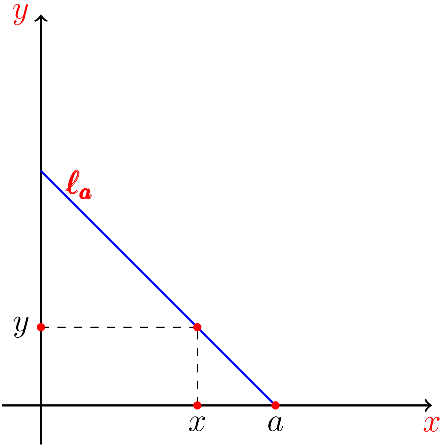
Theorem (R., S.)

If $p > \frac{1}{M}$ ($\dim_{\mathrm{H}} \Lambda > 1$) then for every strictly monotonic smooth function f, $f(\Lambda)$ contains an interval, almost surely conditioned on non-extinction.

Examples:

- $\{x + y : (x, y) \in \Lambda\} \supset$ interval .
- $\{x \cdot y : (x, y) \in \Lambda\} \supset$ interval .

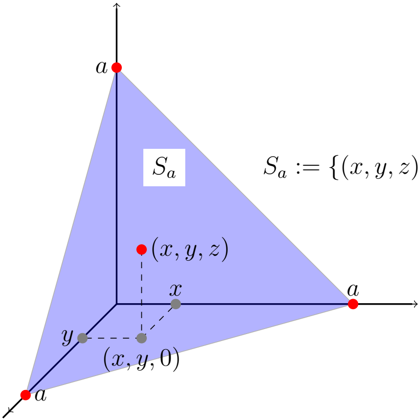
Outline



Similarly, the arithmetic sum

$$\Lambda_1 + \Lambda_2 := \{a : \ell_a \cap \Lambda_1 \times \Lambda_2 \neq \emptyset\} .$$

is the 45° projection of $\Lambda_1 \times \Lambda_2$.



$S_a := \{(x, y, z) : x + y + z = a\}$

$$a = x + y + z \iff (x, y, z) \in S_a$$

$$\Lambda_1 + \Lambda_2 + \Lambda_3 = \{a : S_a \cap \Lambda_1 \times \Lambda_2 \times \Lambda_3 \neq \emptyset\}.$$

Recall:

If $\frac{1}{M^2} < p \leq \frac{1}{M}$ *then for almost all realizations of* Λ *(conditioned on* $\Lambda \neq \emptyset$*) and* ***for all straight lines*** ℓ *: there exists a constant* C *such that* ***the number of level*** n ***squares having nonempty intersection with*** Λ ***is at most*** $c \cdot n$.

The same theorem holds if we substitute the two-dimensional Mandelbrot percolation Cantor set with the product of two one dimensional Cantor sets having the same M and probabilities p_1, p_2 such that $p = p_1 \cdot p_2$.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be one dimensional Mandelbrot percolation fractals constructed with the same M but with may be different probabilities p_1, p_2, p_3. Let Λ be the three dimensional Mandelbrot percolation with the same M and

$$p := p_1 p_2 p_3$$

The random Cantor sets

$$\Lambda_1 \times \Lambda_2 \times \Lambda_3 \text{ and } \Lambda$$

share many common features:

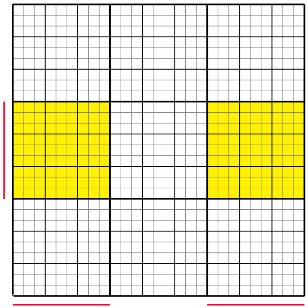
$$\dim \Lambda_1 \times \Lambda_2 \times \Lambda_3 = \dim \Lambda = \frac{\log M^3 p}{\log M}.$$

conditioned on non-extinction.

Dependency in the product set

$$\Lambda_{123} := \Lambda_1 \times \Lambda_2 \times \Lambda_3, \ \Lambda_{12} := \Lambda_1 \times \Lambda_2.$$

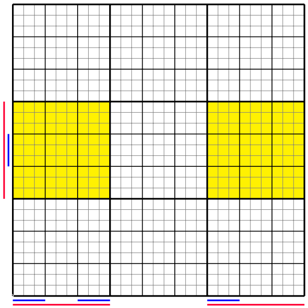
In Λ_{123} and in Λ_{12} there is NO independence between the successors of two cubes having one side common.



Dependency in the product set

$$\Lambda_{123} := \Lambda_1 \times \Lambda_2 \times \Lambda_3, \;\; \Lambda_{12} := \Lambda_1 \times \Lambda_2.$$

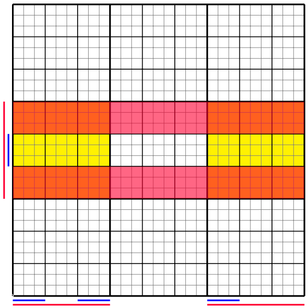
In Λ_{123} and in Λ_{12} there is NO independence between the successors of two cubes having one side common.



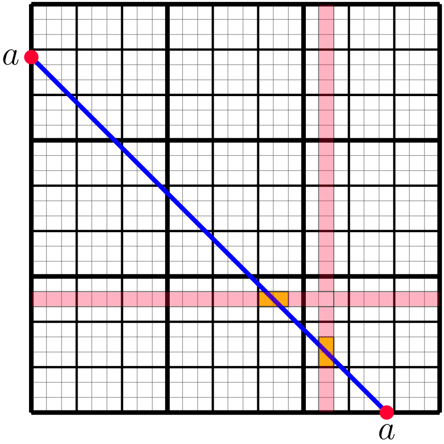
Dependency in the product set

$$\Lambda_{123} := \Lambda_1 \times \Lambda_2 \times \Lambda_3, \;\; \Lambda_{12} := \Lambda_1 \times \Lambda_2.$$

In Λ_{123} and in Λ_{12} there is NO independence between the successors of two cubes having one side common.



Λ and Λ_{12} are a little bit different from the point of 45° projection

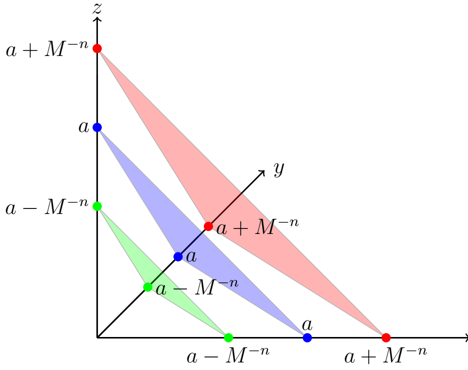


From now we focus on Λ_{123}:

Let $\mathcal{E}^n$ be the set of selected level n cubes in $\Lambda^n_{1,2,3}$. Since $\dim_B \Lambda_{123} > 1$ so for a $\tau > 0$:

$$\#\mathcal{E}^n \approx M^n \cdot M^{\tau \cdot n}.$$

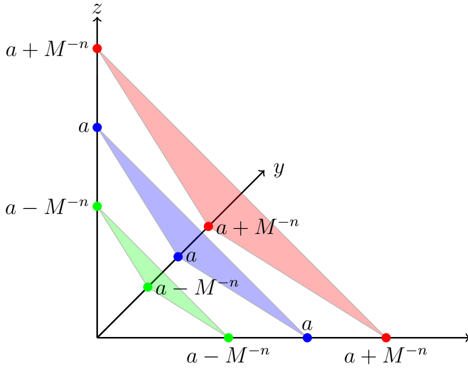
The colored planes: $3M^n$ planes that are orthogonal to $(1,1,1)$ and the consecutive ones are separated by M^{-n}. By pigeon hole principle one of the planes intersects $\text{const} \cdot M^{\tau n}$ selected level n cubes. Assume that this is the **blue plane**.



Let $\mathcal{E}^n$ be the set of selected level n cubes in $\Lambda^n_{1,2,3}$. Since $\dim_B \Lambda_{123} > 1$ so for a $\tau > 0$:

$$\#\mathcal{E}^n \approx M^n \cdot M^{\tau \cdot n}.$$

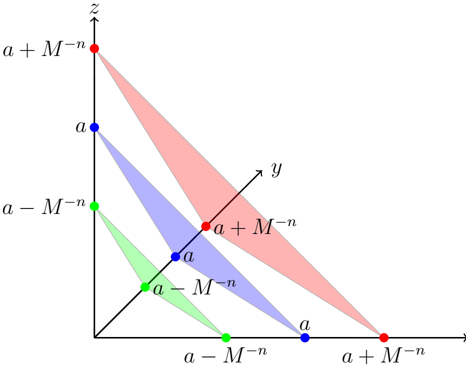
The colored planes: $3M^n$ planes that are orthogonal to $(1,1,1)$ and the consecutive ones are separated by M^{-n}. By pigeon hole principle one of the planes intersects $\text{const} \cdot M^{\tau n}$ selected level n cubes. Assume that this is the **blue plane**.



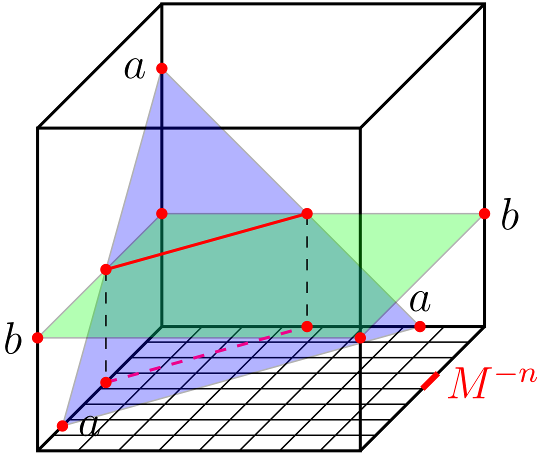
Let $\mathcal{E}^n$ be the set of selected level n cubes in $\Lambda^n_{1,2,3}$.
Since $\dim_B \Lambda_{123} > 1$ so for a $\tau > 0$:

$$\#\mathcal{E}^n \approx M^n \cdot M^{\tau \cdot n}.$$

The colored planes: $3M^n$ planes that are orthogonal to $(1,1,1)$ and the consecutive ones are separated by M^{-n}. By pigeon hole principle one of the planes intersects $\text{const} \cdot M^{\tau n}$ selected level n cubes. Assume that this is the **blue plane**.

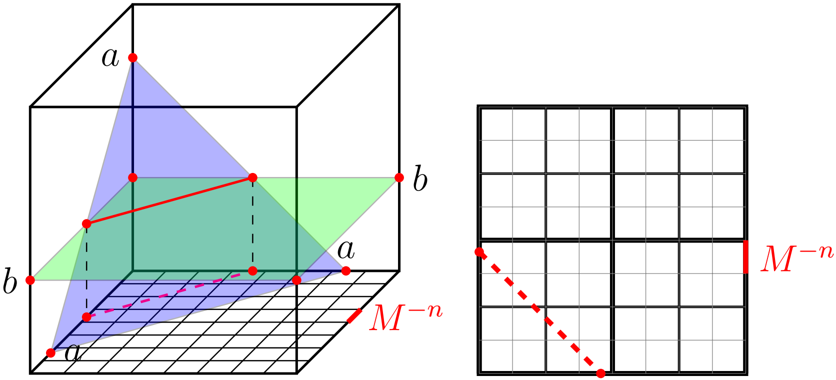


Among the $M^{\tau n}$ cubes which intersect the blue plane the ones sharing one common side are NOT independent. For example those who intersect the red line are NOT independent.



,

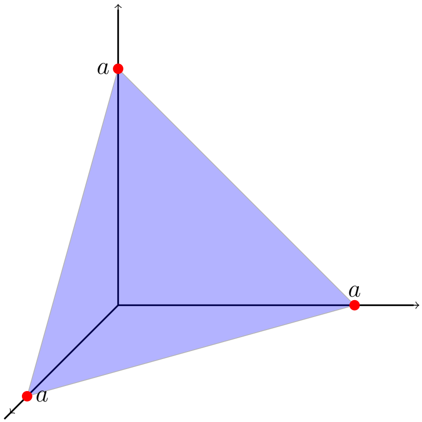
$\dim_H \Lambda_{123} > 1$ but $\dim_H \Lambda_{12}, \dim_H \Lambda_{23}, \dim_H \Lambda_{31} < 1$.



The point is that on the red dashed line there could be potentially M^n selected level n squares but in reality there will be only $c \cdot n$ selected squares.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

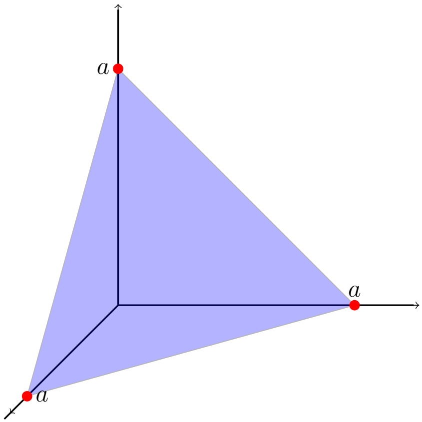
- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

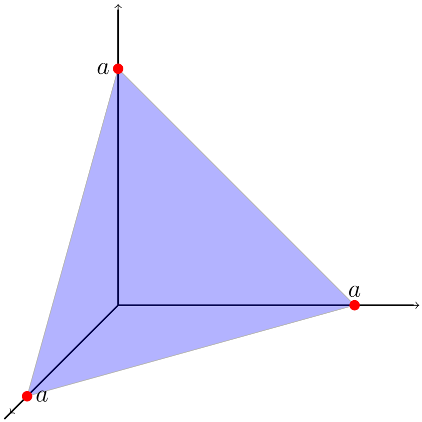
- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

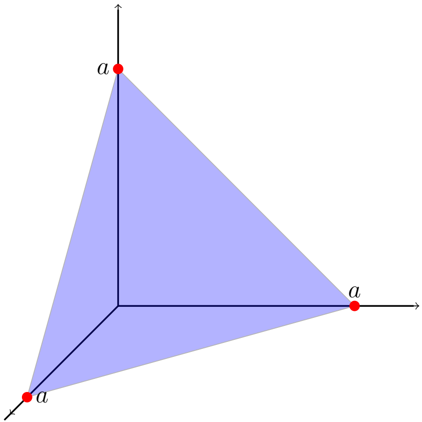
- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

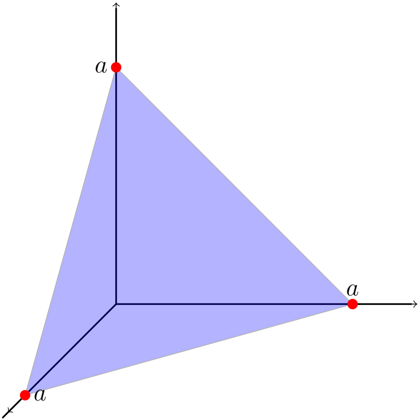
- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

An easy combinatorial Lemma shows that for a $t > 0$ constant there are M^{nt} selected level n squares that have

- no common sides (so what ever happens in these cubes in the future is independent)
- such that they all intersect the blue plane.



Then we use Large deviation theory similarly to Falconer Grimett to get intervals in the projection.

J.T Chayes, L. Chayes, R. Durrett,
Connectivity properties of Mandelbrot's percolation process,
Probab. theory Related Fields , 1988.

F.M. Dekking, and G. R. Grimmett.
Superbranching processes and projections of random Cantor sets.
Probab. Theory Related Fields, 78, (1988), 3, 335–355.

K.J. Falconer, and G.R. Grimmett, On the geometry of random Cantor sets and fractal percolation.
J. Theoret. Probab. Vol. 5. (1992), No.3, 465-485.

B. Mandelbrot,
The fractal geometry of nature,
W.H. Freeman and Co., New York 1983.