Continuity of subadditive pressure

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Self affine sets

- Let $\{f_i(x) = A_i x + t_i\}_{i=1}^m$ be a finite collection contractive affine maps on some Euclidean space $\mathbb{R}^d$. We refer to the A_i as the linear parts and to t_i as the translations.
- It is well known that there exists a unique nonempty compact set $X = X(f_1, \ldots, f_m)$ such that

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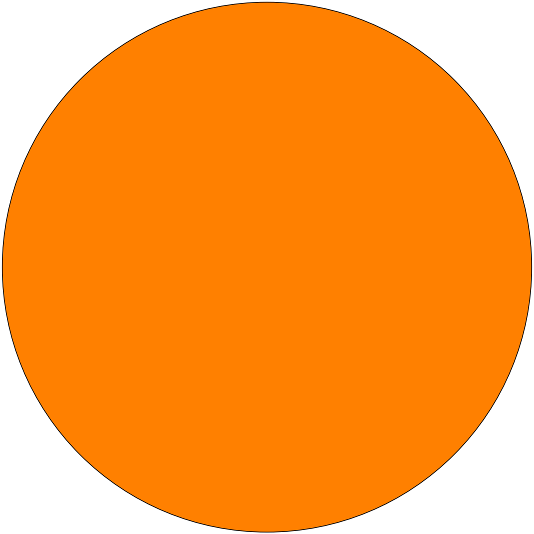

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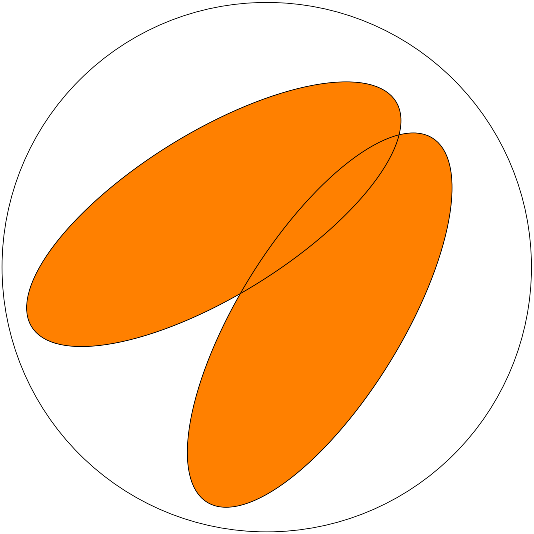

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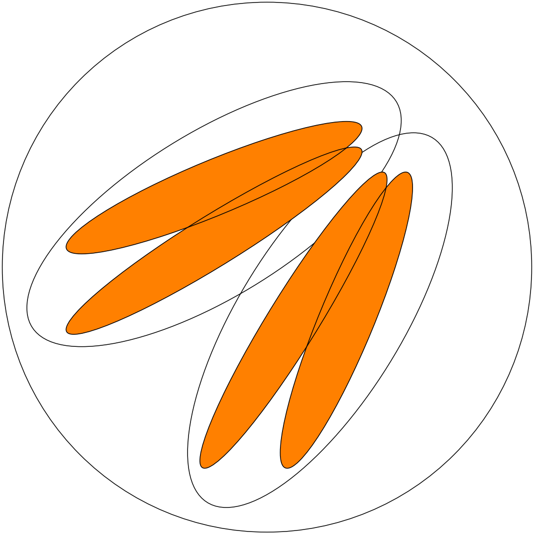

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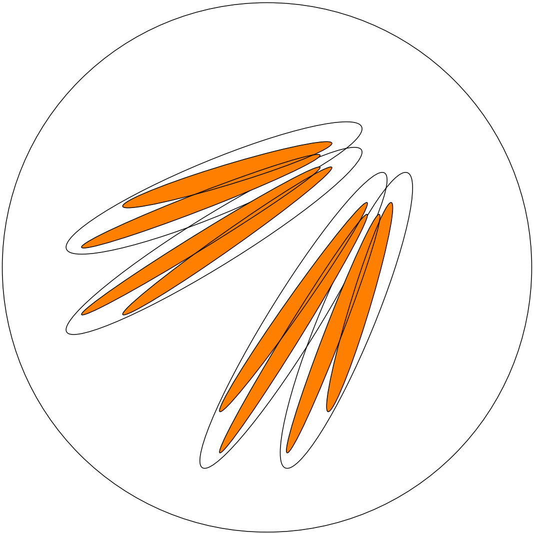

Dimension of self-affine sets: bad news

- ☹ There is no hope of finding a general formula for the dimension of a self-affine set.
- ☹ The Hausdorff and box counting dimensions of a self-similar set may be different (e.g. McMullen carpets).
- ☹ Both the Hausdorff and box counting dimensions are discontinuous as a function of the generating maps.
- ☹ All of the above remains true even if we assume that the pieces $f_i(X)$ are separated (SSC/OSC).

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Subadditive pressure

Let $(A_1 x + t_1, \ldots, A_m x + t_m)$ be a self-affine IFS. There exists a very important pressure function $P(A_1, \ldots, A_m; s)$ with the following properties:

1. It depends on the linear parts of the affine maps and a nonnegative number $s \geq 0$; the translations do not come in.
2. For fixed $A = (A_1, \ldots, A_m)$, $P(A, s)$ is a strictly decreasing function of s. Moreover, $P(A, 0) = \log m > 0$ and $\lim_{s\to\infty} P(A, s) = -\infty$.
3. Hence, there is a unique $s_0 = s_0(A)$ such that $P(A, s_0) = 0$. This value s_0 is known as the singularity, singular value or affinity dimension.

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Singularity dimension and dimension: good news

- ☺ (Douady and Osterle; Falconer) $\dim_H(X) \leq \dim_B(X) \leq s_0$ for all self-affine sets.
- ☺ (Falconer; Solomyak) If the norms of the A_i are $< 1/2$, then for a.e. choice of translation $t_1, \ldots, t_m$, we have

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- ☺ (Falconer; Hueter and Lalley; Käenmäki and S.) There are various explicit conditions on the A_i, t_i which guarantee that the Hausdorff and/or the box counting dimensions of X equal s_0.
- ☺ (Many people) Many generalizations to nonlinear situations, measures (instead of sets), multifractal problems, countably many maps, random settings, etc.

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Summary so far

- ☹ The problem of calculating the dimension of a specific self-affine set is untractable.
- ☺ However, the singularity dimension is in some sense the "expected" value of the Hausdorff/box dimension (it is always an upper bound, it is typically the dimension and also in concrete classes of examples).
- The singularity dimension $s_0(A_1, \ldots, A_m)$ is defined by the condition $P(A_1, \ldots, A_m; s_0) = 0$.

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The question and the result

Question (Folklore, Solomyak, Falconer and Sloan)

Is the singularity dimension continuous as a function of $A_1, \ldots, A_m$? More generally, is the subadditive pressure $P(A_1, \ldots, A_m; s)$ jointly continuous?

Theorem (D-J Feng and P.S.)

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Estimating the Hausdorff measure of X in $\mathbb{R}^2$

In order to estimate the s-dimensional Hasudorff measure of X, we use that

$$X \subset \bigcup_{(i_1 \ldots i_k)} f_{i_1} \cdots f_{i_k}(B).$$

This is a cover of X by ellipses.

We can cover each ellipse by disks separately (this may not be optimal if the ellipses overlap substantially or are aligned in a pattern that makes it better to cover many at once).

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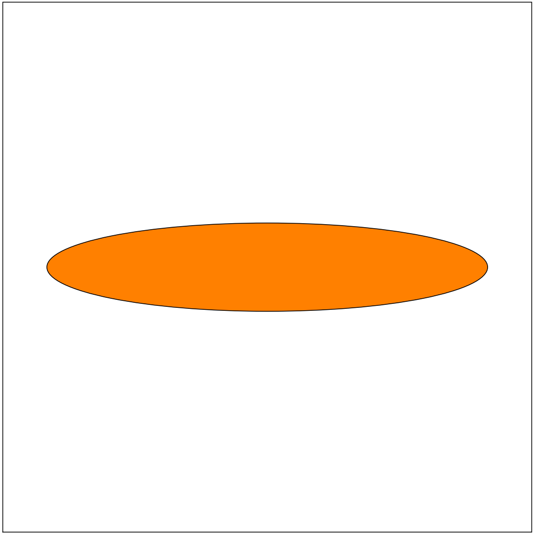

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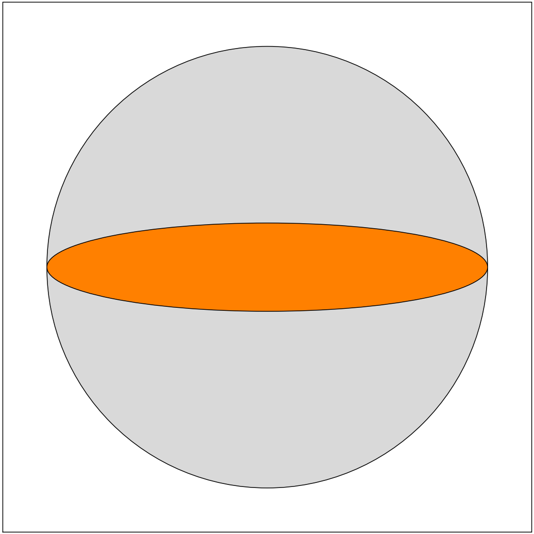

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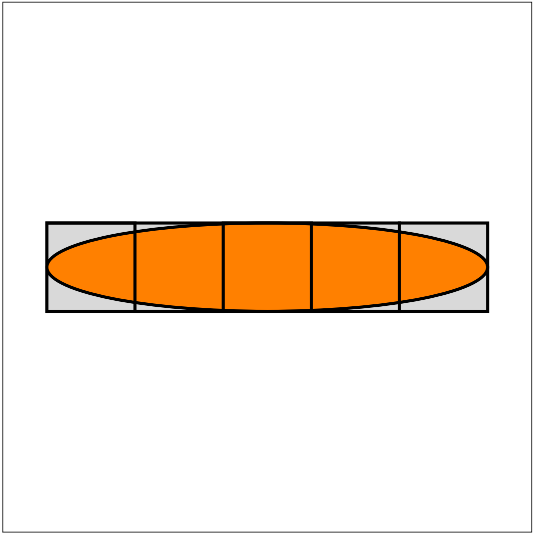

Singular value function

The singular value function (SVF) $\phi^s(A)$ is the contribution to s-dimensional Hausdorff measure of the ellipse $A(B)$

Given $A \in GL_d(\mathbb{R})$, $\alpha_1(A) \geq \cdots \geq \alpha_d(A) > 0$ are the singular values of A (i.e. the semi-axes of the ellipsoid $A(B)$, or the square roots of the eigenvalues of A^*A.)

Then

$$\phi^s(A) = \alpha_1(A) \cdots \alpha_m(A) \alpha_{m+1}^{s-m}.$$

If $d = 2$, then

$$\phi^s(A) = \alpha_1(A)^s \quad \text{if } \lfloor s \rfloor = 1,$$
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Subadditive pressure

Definition

Let $A = (A_1, \ldots, A_m) \in (GL_d(\mathbb{R}))^m$. Given $s \geq 0$, the subadditive topological pressure $P(A, s)$ is defined as

$$P(A, s) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{i_1 \ldots i_n} \phi^s(A_{i_1} \cdots A_{i_n}) \right).$$

Some earlier partial continuity results

Theorem (Folklore, Falconer-Sloan, Käenmäki-S.)

$A \to P(A, s)$ is always upper semicontinuous. Under each of the following assumptions, A is a point of continuity of map $P(\cdot, s)$:

- *$(A_1, \ldots, A_m)$ satisfies certain strong irreducibility condition.*
- *$A_1 = \cdots = A_m$ is an upper triangular map.*
- *All A_i map a projective closed convex set into its interior (cone condition) and $s \leq 1$.*
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Some generalizations

We prove continuity of more general subadditive pressures arising in:

- The study of dimension of certain non-affine, non-conformal repellers,
- The multifractal spectrum of Gibbs measures on self-affine sets,
- Some randomized models of self-affine sets.

Our result also implies that equilibrium measures for $P(A, s)$ are continuous as a function of A.

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Variational principle

Theorem (A. Käenmäki)

Given A, s,

$$P(A,s) = max\left\{h_\mu + \lim_{n\to\infty}\frac{1}{n}\int \log\phi^s(A_{i_1}\cdots A_{i_n})d\mu(\mathbf{i})\right\},$$

where the maximum is over all ergodic measures μ on $\{1,\ldots,m\}^{\mathbb{Z}}$, and h_μ is measure-theoretical entropy.

Definition

A measure μ achieving the maximum is called an equilibrium measure.

Question

Is the set of ergodic equilibrium measures always finite?

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Given an ergodic measure μ, there exist $\lambda_1 > \ldots > \lambda_k$ and $d_1, \ldots, d_k$, such that for μ-almost all $\mathbf{i}$ there exists a measurable decomposition

$$\mathbb{R}^d = \bigoplus_{j=1}^{k} E_j(\mathbf{i})$$

such that for μ-a.e. $\mathbf{i}$,

1. $\dim E_j(\mathbf{i}) = d_j$,
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Oseledets Theorem

Theorem

Given an ergodic measure μ, there exist $\lambda_1 > \ldots > \lambda_k$ and $d_1, \ldots, d_k$, such that for μ-almost all $\mathbf{i}$ there exists a measurable decomposition

$$\mathbb{R}^d = \bigoplus_{j=1}^{k} E_j(\mathbf{i})$$

such that for μ-a.e. $\mathbf{i}$,

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Idea of proof I

- From now on assume $d = 2$ for simplicity (the ideas are the same in higher dimensions, but there are substantial technical issues).
- Suppose μ is an ergodic measure with different Lyapunov exponents $\lambda^+ > \lambda^-$. Write $\mathbb{R}^2 = E^+(\mathbf{i}) \oplus E^-(\mathbf{i})$ for the Oseledets decomposition.
- Key observation: suppose that for some $\mathbf{i}$ and some large n, $E^+(\mathbf{i}) \sim E^+(\sigma^n \mathbf{i})$ and $E^-(\mathbf{i}) \sim E^-(\sigma^n \mathbf{i})$. Then $A_{i_n} \cdots A_{i_1}$ maps a narrow cone around $E^+(\mathbf{i})$ into itself.

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Idea of proof II

- We consider the space $\mathcal{X}$ of all splittings $\mathbb{R}^2 = E^+ \oplus E^-$, which has a natural metric.
- The push-down of the measure μ under the Oseledets splitting is a measure on $\mathcal{X}$. Let $\Sigma = (\widetilde{E}^+, \widetilde{E}^-)$ be a point in the support.
- Let $\mathcal{X}_\varepsilon$ be the ε neighborhood of Σ. By the ergodic theorem (or Poincaré recurrence), for μ-a.e. $\mathbf{i}$ for which the splitting is in $\mathcal{X}_\varepsilon$, there are infinitely many $n \geq 1$ such that the splitting of $\sigma^n \mathbf{i}$ is also in $\mathcal{X}_\varepsilon$.
- By the key remark, when this happens we know that $A_{i_n} \cdots A_{i_1}$ maps the cone $C(\widetilde{E}^+, \varepsilon)$ into its interior.

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Idea of proof III

We know that $\mu\{ß : \mathbf{i} \in \mathcal{X}_\varepsilon\} > 0$. By the ergodic theorem and the previous remarks, we can find arbitrarily large n and a collection of words $I = \{(i_n, \ldots, i_1)\}$ such that:

1. $\sum_{(i_n \ldots i_1) \in I} \mu[i_n \ldots i_1] > c(\varepsilon) > 0$.
2. There is a cone $C(\widetilde{E}^+, \varepsilon)$ which is mapped into its interior by $A_{i_n} \cdots A_{i_1}$ for $(i_n \ldots i_1) \in I$.

It follows that the IFS $\{A_{i_n} \cdots A_{i_1} : (i_n \ldots i_1) \in I\}$ has pressure arbitrarily close to that of the original IFS (after normalization) and satisfies the cone condition. QED.

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The end

Thanks!