Fractional Lévy Processes: Paths, Dimensions, and Related

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Abstract and Outline

In this talk, we report some progress for paths and dimensions of the harmonizable and the linear fractional processes driven by a non-Gaussian Lévy process with, say, exponential moment. We show that these two processes are different in law, due to different modulus of continuity. We present a dimension formula of $X(E)$ in the harmonizable case. We propose the multi-fractional and the exponential processes associated with such fractional processes. We also propose the relativistic stable motion, that is, the subordination of BM by a relativistic stable-subordinator, as a good candidate for such driving LP, since it meets the large and the small scaling features of fLP, which are lack for fBM. This talk is adapted from joint works with several co-authors; confer to URL: `http://www.math.ntu.edu.tw/~shiehnr` .

From fractional BM to fractional LP

- fBM: a real-valued centered Gaussian process $X = \{X(t), \quad t \in \mathbb{R}\}$, with stationary increments and with the covariance determined by
- $E\left[|X(t) - X(s)|^2\right] = const \cdot |t - s|^{2H}$.
- $H = 1/2$ it is BM; $1/2 < H < 1$ it has long-range dependence.
- fBM has two well-known *equivalent in the law, up to constants*, representations: the linear and the (real) harmonizable fBMs
- $\Delta(t) = \int_{-\infty}^{\infty}(|t-u|^{H-1/2} - |u|^{H-1/2})dB(u)$.
- $\Psi(t) = Re \int_{-\infty}^{\infty} \frac{e^{it\lambda}-1}{i\lambda}|\lambda|^{1-H-1/2}d\tilde{B}(\lambda)$.
- To catch the *non-Gaussianity* for, say, stylized facts in financial economics; Benassi *et al.* (2002,2004) define two fractional processes, using the driving force to be a two-sided symmetric Lévy process with second moment to replace BM in the above two representations, assuming that $1/2 < H < 1$.
- The fLPs are featured by the large and the small scaling asymptotic, which the fBM are lack of.

The linear and the harmonizable fLPs are not equivalent

- A natural question: are the two fLPs so-defined equivalent (in the law, up to constants) ? This questions can be traced back to works on stable-motion as the driving force.
- The two fLPs are both of continuous-paths, of stationary increments, and of the same covariance as the fBM (up to constant). Thus, this question is not obvious at glance.
- Jointly with M. Maejima, it is proved (2011) that the two fLPs are **not** equivalent.
- The method is simple yet some interesting, we show that the two fLPs has different modulus of continuity.
- An early result of Billingsley (1974) tells that the two fLPs then cannot be law-equivalent.
- Using different paths properties to distinguish different stochastic processes seems of some potential further usage.

Dimension formula of $X(E)$

- Given a separable stochastic process, one or multi parameter, it is of interesting of determine the dimension (Hausdorff, packing, and others) of $X(E)$, E being a subset of the parameter space.
- Jointly with Y. Xiao, it is proved (2010) that, among others, for the harmonizable fLP,
- $\dim X(E) = \min\{1, (1/H)\dim E\}$; this is the same as the fBM case.
- It is tempting to see that this formula should hold for linear fLP; however it is not obvious, since in the linear fLP it appears a equally guess-able candidate, and this comes from the small scale limit for fLP is not fBM, unlike the harmonizable fLP.

Relativistic LP: a good driving force

- The driving LP can be anyone with Lévy measure $\nu(dx) \in L^p, \forall p \geq 2$. We propose to use a LP with generator $H_{\alpha,m} := m - (m^{\frac{2}{\alpha}} - \Delta)^{\frac{\alpha}{2}}$, $0 < m, 0 < \alpha < 2$. This is referred as an relativistic α-stable motion. This LP is the subordination of BM by a relativistic α-stable subordinator.
- it has sharp estimate for the transition density function, Z. Chen *et al.* (several works). The Lévy measure $\nu(dx)$ can be explicitly written down.
- it has a large scale asymptotic, dominated by m and a small scale asymptotic, dominated by α. This is used to discuss the multi-scaling limit of the associated PDE (2012), jointly with G. Liu.
- the pseudo-diff op. $H_{\alpha,m}$ can be used to construct a certain " free field", corresponding to the one from the classical Laplacian (2012).

The multi-fractional and the exponential process

- Replace the constant H is the two representations of fBM by a $(0,1)$-valued continuous function $H(t)$, it can be constructed the multi-fractional BM, and this $H(t)$ is then the Hölder exponent function of the process; Jaffard *et al.* (several works). The linear and the harmonizable mfBM are different in law (Stoev and Taqqu, 2005). The level set of a harmonizable mfBM is shown to have a certain multi-fractal property (2011), jointly with A. Ayache and Y. Xiao.
- Consider the stationary OU or CARMA (p, q) process driven by a fBM, its associated exponential process can generate a certain multi-fractal infinite-product process (2009, 2012), jointly with M. Matsui.
- It would interesting to consider the mfLP, and OU or CARMA process driven by a fLP. In the above mentioned works, the Gaussian property of fBM is much used; some necessary estimates seem can go over for the driving LP being, say, the relativistic stable-motion (on-going projects).

Possible implications in Finance

- Two survey papers by Rama Cont (2001, 2011); the first one mentions the usage of multi-fractal process for the stylized facts of financial time-series, and the second one mentions the financial time-series under the influence of limit order book (ultra-high frequency transactions).
- Some, likely illusive, relation of mfBM/mfLP and the exponential of fractionally-driven OU/CARMA process to Finance ?

Thank You !