Hardy-Littlewood series and (even) continued fractions

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joint work with T. Rivoal (CNRS, Grenoble)

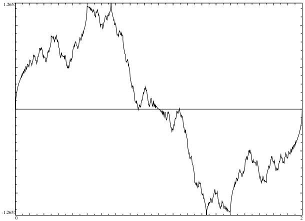
Advances on Fractals and Related Fields

The Chinese University of Hong-Kong

1 - Introduction

Non-differentiable Riemann function:

$$R_2(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{k^2}$$



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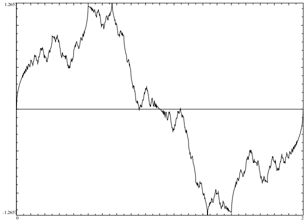

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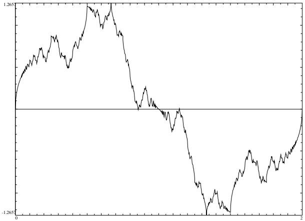

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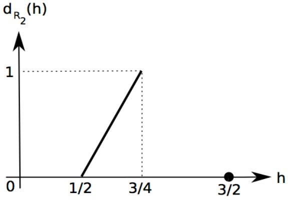
Local Hölder exponent of a L^∞-function f: When $h_f(x) < 1$,

$$h_f(x) = \liminf_{h \to 0^+} \frac{\log |f(x+h) - f(x)|}{\log h}$$

(when f is differentiable, introduce a Taylor polynomial)

Multifractal Spectrum of R_2 (Jaffard, 1999):



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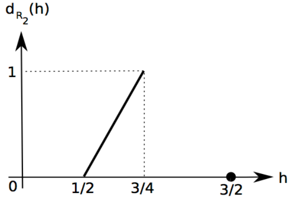

Idea: • Use the wavelet $\psi(x) = (x+i)^{-2}$ and compute the wavelet transform of R_2:

$$W_{R_2}(a,b) = \frac{1}{a}\int_{\mathbb{R}} R_2(x)\psi\left(\frac{x-b}{a}\right)dx$$

and prove (graduate-level complex analysis) that

$$W_{R_2}(a,b) = a\,(2\cdot\theta(b+ia)-1),$$

where $\theta(z) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 z}$ is the Theta Jacobi function.

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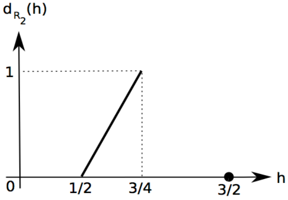

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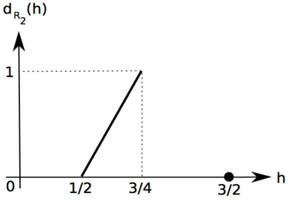

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• Use the Theta group ($\theta(z+2) = \theta(z)$ and $\theta(-1/z) = \theta(z)$) to study $W_{R_2}(a,b)$ when $a \to 0^+$.

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Theorem (F. Chamizo and A. Ubis, preprint 2012)

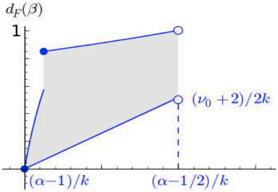
Let

$$R_s^P(x) = \sum_{n=1}^{+\infty} \frac{e^{i\pi P(n)x}}{n^s},$$

where P is of degree k, then if $1 + k/2 < s < k$ one has

$$(\nu_0+2)\beta \le d_{R_s^P}\left(\beta + \frac{\alpha-1}{k}\right) \le \begin{cases} \frac{2\beta}{2^{-k}+\beta} & \text{if } 0 \le \beta < \frac{1}{k2^{-k}} \\ \frac{3}{2} - \sqrt{\frac{k+4}{4k} - 2\beta} & \text{if } \frac{1}{k2^{-k}} \le \beta < \frac{1}{2k}, \end{cases}$$

where ν_0 is the greatest multiplicity of the zeros of P'.



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  - Convergence?
  - Local regularity? (distinguish the points)
  - Exploit the modular forms to rewrite $F_s(x,t)$ in a more explicit form in terms of the Diophantine properties of x (more precisely in terms of the even continued fraction expansion).

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Let $x = (P_k/Q_k)_{k\geq 0}$ *(its continued fraction) be an irrational number in* $(0,1)$*, and let* $t \in \mathbb{R}$*.*

(i) If $s \in (\frac{1}{2}, 1)$*, then* $F_s(x,t)$ *is absolutely convergent when*

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Hence, if $\mu(x) = \sup\left\{\mu \geq 1 : \left|x - \frac{p}{q}\right| < \frac{1}{q^{1+\mu}} \text{ for i.m. } q \geq 1\right\}$, then

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- $F_1(\cdot,t)$ does not converge only on a subset of the Liouville numbers (dimension 0).

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The modular nature of $F_s(x,t)$ implies that the map of $[-1,1] \setminus \{0\}$ given by

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Just for fun: the function $\Omega_s(x,t)$ is $\Omega_s(x,t) = \begin{cases} I_s(x,t) & \text{when } x>0 \\ \overline{I_s(-x,-t)} & \text{when } x<0 \end{cases}$,

where:

$$I_s(x,t) = \int_{1/2-\rho\infty}^{1/2+\rho\infty} \frac{e^{i\pi z^2 x + 2i\pi z\{t\}}}{z^s(1-e^{2i\pi z})} \mathrm{d}z$$

$$+ \rho x^s \int_{-\infty}^{\infty} e^{-\pi x u^2} \left(\sum_{k=1}^{\infty} e^{-i\pi(k-\{t\})^2/x} \left(\frac{1}{(\rho x u + k - \{t\})^s} - \frac{1}{k^s} \right)\right) \mathrm{d}u.$$

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Important: $\sigma(x)$ is the sign of x.

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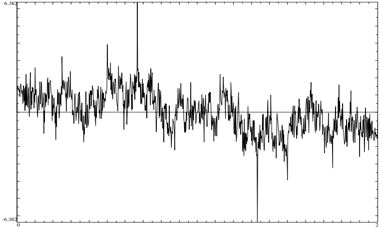
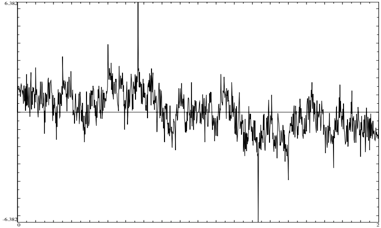


Plot of $\text{Im}(F_{0.7,100}(x)) = \sum_{k=1}^{100} \frac{\sin(\pi k^2 x)}{k^{0.7}}$ on $[0, 2]$



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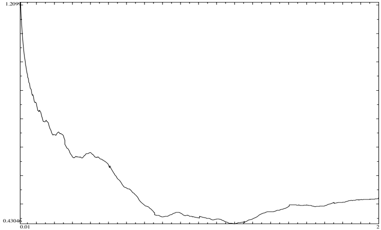

Plot of $\operatorname{Im}(F_{0.7,1000}(x) - e^{i\pi/4} x^{0.2} F_{0.7,\lfloor 1000x \rfloor}(-1/x))$ on $[0, 2]$

4 - Even continued fractions

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Starting with a given integer n, then the integer $\lfloor \lfloor \cdots \lfloor \lfloor n|x| \rfloor |T(x)| \rfloor \cdots \rfloor |T^\ell(x)| \rfloor$ tends to zero, and we get an empty sum.

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At the end, one gets

$$F_{s,n}(x) = \sum_{j=0}^{K(n,x)} e^{i\frac{\pi}{4}\sum_{\ell=0}^{j-1}\sigma(T^\ell x)} |xT(x)\cdots T^{j-1}(x)|^{s-\frac{1}{2}}\,\Omega_s\big(T^j(x)\big)$$

for some integer $K(n,x)$ that tends to infinity when n tends to infinity.

Theorem

Let $s \in (\frac{1}{2}, 1)$. If $x \in (-1, 1)$ is an irrational number such that

$$\sum_{j=0}^{\infty} \frac{|xT(x)\cdots T^{j-1}(x)|^{s-\frac{1}{2}}}{|T^j(x)|^{\frac{1-s}{2}}} < \infty,$$

then $F_s(x)$ is also convergent and the following identity holds:

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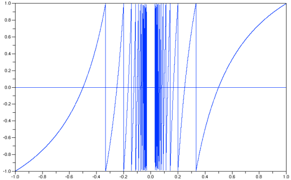
$([-1,1],T)$ is a dynamical system with parabolic points -1 and 1.

Infinite ergodic measure.



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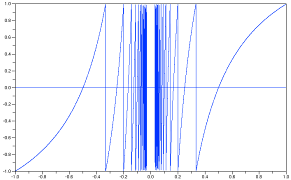

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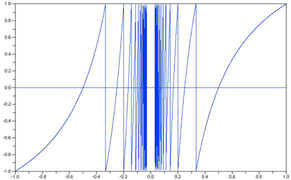

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Proposition

x has a unique even continued fraction (ECF) expansion $x = \cfrac{e_1}{a_1 + \cfrac{e_2}{a_2 + \cfrac{e_3}{a_3 + \ldots}}}$,

- a_j *the unique* **even** *integer such that* $T^j(x) - a_j \in (-1,1)$
- $e_j = \sigma(T^j(x)) \in \{-1,1\}$.

Schweiger, Kraaikamp, Lopes, Sinai (and students)...

We define the n-th convergent and the n-th remainder respectively as

$$\frac{p_n}{q_n} := \cfrac{1}{a_1 + \cfrac{e_1}{a_2 + \cfrac{e_2}{\ddots + \cfrac{e_{n-1}}{a_n}}}} \quad \text{and} \quad x_n := \cfrac{e_n}{a_{n+1} + \cfrac{e_{n+1}}{a_{n+2} + \cfrac{e_{n+2}}{\ddots}}}.$$

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ECF expansions are obtained from the classical expansions via an iterative method: for any positive integers (A, B, C) and any $\gamma \geq 0$, observe that

$$A + \cfrac{1}{B + \cfrac{1}{C+\gamma}} = (A+1) + \cfrac{-1}{2 + \cfrac{-1}{2 + ... + \cfrac{-1}{2 + \cfrac{-1}{(C+1)+\gamma}}}},$$

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From $x := \cfrac{1}{A_1 + \cfrac{1}{A_2 + \cfrac{1}{\ddots + \cfrac{1}{A_n + ...}}}}$, we apply the *singularization* each time we have an odd A_n.

If all the A_n's are even, then this expansion is indeed the ECF of x.

Proposition

For every irrational $x \in [0,1]$ *and every* $j \geq 1$, *we have*

$$q_{n+1} > q_n, \quad \lim_{n \to +\infty} (q_{n+1} - q_n) = +\infty$$

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The series

$$\sum_{n\geq 1} |xT(x)\cdots T^n(x)|^\alpha$$

may diverge (Aaronson, Sinai and students studied convergence in probability), while

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always converges, since $|xG(x)\cdots G^n(x)| \leq \dfrac{1}{Q_n}$.

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Let Ω be a bounded function, differentiable at 1 and -1. The series

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converges if $\displaystyle\sum_{n=1}^{\infty} \frac{Q_{n+1}}{Q_n^{\alpha+1}} < \infty$ *(i.e. when $\mu(x) \leq 1+\alpha$).*

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For any $\alpha > 0$ and $\beta \geq 0$, and any irrational number $x \in (0,1)$, the series

$$\sum_{j=0}^{\infty} \frac{|xT(x)\cdots T^{j-1}(x)|^{\alpha}}{|T^j(x)|^{\beta}}$$

converges if $\displaystyle\sum_{n=1}^{\infty} \frac{Q_{n+1}^{\beta+1}}{Q_n^{\alpha+\beta+1}} < \infty$ $\left(\text{i.e. when } \mu(x) \leq 1 + \frac{\alpha}{\beta+1}\right)$.

$$\mu(x) = \sup\left\{\mu \geq 1 : \left|x - \frac{p}{q}\right| < \frac{1}{q^{1+\mu}} \text{ for infinitely many integers } q \geq 1\right\}.$$

Theorem

Let Ω be a bounded function, differentiable at 1 and -1. The series

$$\sum_{j=1}^{\infty} |xT(x)\cdots T^{j-1}(x)|^{\alpha}\ \Omega\big(T^j(x)\big)\, e^{i\frac{\pi}{4}\sum_{\ell=0}^{j-1}\sigma(T^{\ell}x)}$$

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Solution: Only a technical detail in the proof forces us to ensure absolute convergence of the sum $\displaystyle\sum_{j=0}^{\infty} e^{i\frac{\pi}{4}\sum_{\ell=0}^{j-1}\sigma(T^\ell x)} |xT(x)\cdots T^{j-1}(x)|^{s-\frac{1}{2}}$. If we could replace it with the simple convergence, then we would be optimal.

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Let's come back to R_2:

$$R_2(x) = \sum_{k=1}^{\infty} \frac{\sin(\pi k^2 x)}{k^2} = \operatorname{Im}\left(\sum_{j=0}^{\infty} e^{i\frac{\pi}{4}\sum_{\ell=0}^{j-1}\sigma(T^\ell x)} |xT(x)\cdots T^{j-1}(x)|^{\frac{3}{2}}\, \Omega_2\big(T^j(x)\big) \right),$$

where Ω_2 is differentiable (except at 0).

The use of T instead of G explains why the regularity depends on the approximation rate by rationals p/q with p, q both odd.

5. Questions

- Find some courage to finish the theorem...

- Use the modular expression to completely characterize the multifractal properties of R_s, for $s > 1$.

- Distinguish, for R_s with $1/2 < s \leq 1$, the different local behaviors according to the Diophantine exponent.

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- Apply the same techniques to other functions.

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